This section continues the discussion of secants and tangents begun in Section 2.1. We calculate limits of secant slopes to find tangents to curves.

**What Is a Tangent to a Curve?**

For circles, tangency is straightforward. A line \( L \) is tangent to a circle at a point \( P \) if \( L \) passes through \( P \) perpendicular to the radius at \( P \) (Figure 2.63). Such a line just *touches*
the circle. But what does it mean to say that a line $L$ is tangent to some other curve $C$ at a point $P$? Generalizing from the geometry of the circle, we might say that it means one of the following:

1. $L$ passes through $P$ perpendicular to the line from $P$ to the center of $C$.
2. $L$ passes through only one point of $C$, namely $P$.
3. $L$ passes through $P$ and lies on one side of $C$ only.

Although these statements are valid if $C$ is a circle, none of them works consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect $C$ at other points or cross $C$ at the point of tangency (Figure 2.64).

To define tangency for general curves, we need a dynamic approach that takes into account the behavior of the secants through $P$ and nearby points $Q$ as $Q$ moves toward $P$ along the curve (Figure 2.65). It goes like this:

1. We start with what we can calculate, namely the slope of the secant $PQ$.
2. Investigate the limit of the secant slope as $Q$ approaches $P$ along the curve.
3. If the limit exists, take it to be the slope of the curve at $P$ and define the tangent to the curve at $P$ to be the line through $P$ with this slope.

This approach is what we were doing in the falling-rock and fruit fly examples in Section 2.1.
EXAMPLE 1  Tangent Line to a Parabola

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution  We begin with a secant line through $P(2, 4)$ and nearby. We then write an expression for the slope of the secant $PQ$ and investigate what happens to the slope as $Q$ approaches $P$ along the curve:

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} = \frac{h^2 + 4h}{h} = h + 4.$$  

If $h > 0$, then $Q$ lies above and to the right of $P$, as in Figure 2.66. If $h < 0$, then $Q$ lies to the left of $P$ (not shown). In either case, as $Q$ approaches $P$ along the curve, $h$ approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} (h + 4) = 4.$$  

We take 4 to be the parabola’s slope at $P$. The tangent to the parabola at $P$ is the line through $P$ with slope 4:

$$y = 4 + 4(x - 2) \quad \text{Point-slope equation}$$

$$y = 4x - 4.$$ 

![FIGURE 2.66  Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).](image)

Finding a Tangent to the Graph of a Function

The problem of finding a tangent to a curve was the dominant mathematical problem of the early seventeenth century. In optics, the tangent determined the angle at which a ray of light entered a curved lens. In mechanics, the tangent determined the direction of a body’s motion at every point along its path. In geometry, the tangents to two curves at a point of intersection determined the angles at which they intersected. To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the same dynamic procedure. We calculate the slope of the secant through $P$ and a point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.67). If the limit exists, we call it the slope of the curve at $P$ and define the tangent at $P$ to be the line through $P$ having this slope.
2.7 Tangents and Derivatives

DEFINITIONS Slope, Tangent Line

The slope of the curve \( y = f(x) \) at the point \( P(x_0, f(x_0)) \) is the number

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

(provided the limit exists).

The tangent line to the curve at \( P \) is the line through \( P \) with this slope.

Whenever we make a new definition, we try it on familiar objects to be sure it is consistent with results we expect in familiar cases. Example 2 shows that the new definition of slope agrees with the old definition from Section 1.2 when we apply it to nonvertical lines.

EXAMPLE 2 Testing the Definition

Show that the line \( y = mx + b \) is its own tangent at any point \( (x_0, mx_0 + b) \).

Solution We let \( f(x) = mx + b \) and organize the work into three steps.

1. Find \( f(x_0) \) and \( f(x_0 + h) \).

\[
f(x_0) = mx_0 + b
f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b
\]

2. Find the slope \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \).

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}
= \lim_{h \to 0} \frac{mh}{h} = m
\]

3. Find the tangent line using the point-slope equation. The tangent line at the point \( (x_0, mx_0 + b) \) is

\[
y = (mx_0 + b) + m(x - x_0)
y = mx_0 + b + mx - mx_0
y = mx + b.
\]

Let’s summarize the steps in Example 2.

Finding the Tangent to the Curve \( y = f(x) \) at \( (x_0, y_0) \)

1. Calculate \( f(x_0) \) and \( f(x_0 + h) \).
2. Calculate the slope

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.
\]
3. If the limit exists, find the tangent line as

\[
y = y_0 + m(x - x_0).
\]
EXAMPLE 3  Slope and Tangent to $y = 1/x$, $x \neq 0$

(a) Find the slope of the curve $y = 1/x$ at $x = a \neq 0$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as $a$ changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{1}{a + h} - \frac{1}{a} = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{a + h} - \frac{1}{a} \right) = \lim_{h \to 0} \frac{-h}{ha(a + h)} = \lim_{h \to 0} \frac{-1}{a(a + h)} = -\frac{1}{a^2}.
$$

Notice how we had to keep writing “\( \lim_{h \to 0} \)” before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number $a$ may be positive or negative, but not 0.

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$
-\frac{1}{a^2} = -\frac{1}{4}.
$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.68).

(c) Notice that the slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 2.69). We see this situation again as $a \to 0^-$. As $a$ moves away from the origin in either direction, the slope approaches $0^-$ and the tangent levels off.

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Rates of Change: Derivative at a Point

The expression

\[ \frac{f(x_0 + h) - f(x_0)}{h} \]

is called the difference quotient of \( f \) at \( x_0 \) with increment \( h \). If the difference quotient has a limit as \( h \) approaches zero, that limit is called the derivative of \( f \) at \( x_0 \). If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where \( x = x_0 \). If we interpret the difference quotient as an average rate of change, as we did in Section 2.1, the derivative gives the function’s rate of change with respect to \( x \) at the point \( x = x_0 \). The derivative is one of the two most important mathematical objects considered in calculus. We begin a thorough study of it in Chapter 3. The other important object is the integral, and we initiate its study in Chapter 5.

EXAMPLE 4  Instantaneous Speed (Continuation of Section 2.1, Examples 1 and 2)

In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell \( y = 16t^2 \) feet during the first \( t \) sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock’s speed at the instant \( t = 1 \). Exactly what was the rock’s speed at this time?

Solution  We let \( f(t) = 16t^2 \). The average speed of the rock over the interval between \( t = 1 \) and \( t = 1 + h \) seconds was

\[ \frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2). \]

The rock’s speed at the instant \( t = 1 \) was

\[ \lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}. \]

Our original estimate of 32 ft/sec was right.

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, the limit of the difference quotient, and the derivative of a function at a point. All of these ideas refer to the same thing, summarized here:

1. The slope of \( y = f(x) \) at \( x = x_0 \)
2. The slope of the tangent to the curve \( y = f(x) \) at \( x = x_0 \)
3. The rate of change of \( f(x) \) with respect to \( x \) at \( x = x_0 \)
4. The derivative of \( f \) at \( x = x_0 \)
5. The limit of the difference quotient, \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \)