EXERCISES 4.8

Finding Antiderivatives
In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$
   b. $x^2$
   c. $x^2 - 2x + 1$
2. a. $6x$
   b. $x^3$
   c. $x^7 - 6x + 8$
3. a. $-3x^{-4}$
   b. $x^{-4}$
   c. $x^{-4} + 2x + 3$
4. a. $2x^{-3}$
   b. $\frac{x^{-3}}{2} + x^2$
   c. $-x^{-3} + x - 1$
5. a. $\frac{1}{x^2}$
   b. $\frac{5}{x^2}$
   c. $2 - \frac{5}{x^2}$
6. a. $-\frac{2}{x^3}$
   b. $\frac{1}{2x^3}$
   c. $x^3 - \frac{1}{x^3}$
7. a. $\frac{3}{2} \sqrt{x}$
   b. $\frac{1}{2} \sqrt{x}$
   c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a. $\frac{4}{3} \sqrt{-x}$
   b. $\frac{1}{3} \sqrt{x}$
   c. $\sqrt{-x} + \frac{1}{\sqrt{x}}$
9. a. $\frac{2}{3} x^{-1/3}$
   b. $\frac{1}{3} x^{-2/3}$
   c. $-\frac{1}{3} x^{-4/3}$
10. a. $\frac{1}{2} x^{-1/2}$
    b. $-\frac{1}{2} x^{-3/2}$
    c. $-\frac{3}{2} x^{-5/2}$
11. a. $-\pi \sin \pi x$
    b. $3 \sin x$
    c. $\sin \pi x - 3 \sin 3x$
12. a. $\pi \cos \pi x$
    b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$
    c. $\cos \frac{\pi x}{2} + \pi \cos x$
13. a. $\sec^2 x$
    b. $\frac{2}{3} \sec^2 \frac{x}{3}$
    c. $-\sec^2 \frac{3x}{2}$
14. a. $\csc^2 x$
    b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$
    c. $1 - 8 \csc^2 2x$
15. a. $\csc x \cot x$
    b. $-\csc 5x \cot 5x$
    c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
16. a. $\sec x \tan x$
    b. $4 \sec 3x \tan 3x$
    c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$

Finding Indefinite Integrals
In Exercises 17–54, find the most general antiderivative or indefinite integral. Check your answers by differentiation.

17. $\int (x + 1) \, dx$
18. $\int (5 - 6x) \, dx$
19. $\int \left(3t^2 + \frac{t}{2}\right) \, dt$
20. $\int \left(\frac{t^2}{2} + 4t^3\right) \, dt$
21. $\int (2x^3 - 5x + 7) \, dx$
22. $\int (1 - x^2 - 3x^5) \, dx$
23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) \, dx$
24. $\int \left(\frac{1}{2} - \frac{2}{x^3} + 2x\right) \, dx$
25. $\int x^{-1/3} \, dx$
26. $\int x^{-5/4} \, dx$
27. $\int (\sqrt{x} + \sqrt{x}) \, dx$
28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) \, dx$
29. $\int \left(8y - \frac{2}{y^{1/4}}\right) \, dy$
30. $\int \left(\frac{1}{7} - \frac{1}{y^{3/4}}\right) \, dy$
31. $\int 2x(1 - x^{-3}) \, dx$
32. $\int x^3(x + 1) \, dx$
33. $\int \frac{t^2 + \sqrt{t}}{t^2} \, dt$
34. $\int \frac{4 + \sqrt{t}}{t^2} \, dt$
35. $\int (-2 \cos t) \, dt$
36. $\int (-5 \sin t) \, dt$
37. $\int 7 \sin \frac{\theta}{5} \, d\theta$
38. $\int 3 \cos 5\theta \, d\theta$
39. $\int (-3 \csc^2 x) \, dx$
40. $\int \left(-\frac{\sec^2 x}{3}\right) \, dx$
41. $\int \frac{\csc \theta \cot \theta}{2} \, d\theta$
42. $\int \frac{2}{5} \sec \theta \tan \theta \, d\theta$
43. \( \int (4 \sec x \tan x - 2 \sec^2 x) \, dx \)
44. \( \int \frac{1}{2} (\csc^2 x - \csc x \cot x) \, dx \)
45. \( \int \sin 2x - \csc^2 x \, dx \)
46. \( \int (2 \cos 2x - 3 \sin 3x) \, dx \)
47. \( \int \frac{1 + \cos 4t}{2} \, dt \)
48. \( \int \frac{1 - \cos 6t}{2} \, dt \)
49. \( \int (1 + \tan^2 \theta) \, d\theta \)
50. \( \int (2 + \tan^2 \theta) \, d\theta \)
(Hint: \( 1 + \tan^2 \theta = \sec^2 \theta \))
51. \( \int \cot^2 x \, dx \)
52. \( \int (1 - \cot^2 x) \, dx \)
(Hint: \( 1 + \cot^2 x = \csc^2 x \))
53. \( \int \cos \theta (\tan \theta + \sec \theta) \, d\theta \)
54. \( \int \frac{\csc \theta}{\csc \theta - \sin \theta} \, d\theta \)

### Checking Antiderivative Formulas
Verify the formulas in Exercises 55–60 by differentiation.

55. \( \int (7x - 2)^3 \, dx = \frac{(7x - 2)^4}{28} + C \)
56. \( \int (3x + 5)^{-2} \, dx = -\frac{(3x + 5)^{-1}}{3} + C \)
57. \( \int \sec^2 (5x - 1) \, dx = \frac{1}{5} \tan (5x - 1) + C \)
58. \( \int \csc^2 \left( \frac{x - 1}{3} \right) \, dx = -3 \cot \left( \frac{x - 1}{3} \right) + C \)
59. \( \int \frac{1}{(x + 1)^2} \, dx = -\frac{1}{x + 1} + C \)
60. \( \int \frac{1}{x(x + 1)^2} \, dx = \frac{x}{x + 1} + C \)

61. Right, or wrong? Say which for each formula and give a brief reason for each answer.
   a. \( \int x \sin x \, dx = \frac{x^2}{2} \sin x + C \)
   b. \( \int x \sin x \, dx = -x \cos x + C \)
   c. \( \int x \sin x \, dx = -x \cos x + \sin x + C \)

62. Right, or wrong? Say which for each formula and give a brief reason for each answer.
   a. \( \int \tan \theta \sec^2 \theta \, d\theta = \frac{\sec^3 \theta}{3} + C \)
   b. \( \int \tan \theta \sec^2 \theta \, d\theta = \frac{1}{2} \tan^2 \theta + C \)
   c. \( \int \tan \theta \sec^2 \theta \, d\theta = \frac{1}{2} \sec^2 \theta + C \)

63. Right, or wrong? Say which for each formula and give a brief reason for each answer.
   a. \( \int (2x + 1)^2 \, dx = \frac{(2x + 1)^3}{3} + C \)
   b. \( \int 3(2x + 1)^2 \, dx = (2x + 1)^3 + C \)
   c. \( \int 6(2x + 1)^2 \, dx = (2x + 1)^3 + C \)

64. Right, or wrong? Say which for each formula and give a brief reason for each answer.
   a. \( \int \sqrt{2x + 1} \, dx = \sqrt{x^2 + x} + C \)
   b. \( \int \sqrt{2x + 1} \, dx = \sqrt{x^2 + x} + C \)
   c. \( \int \sqrt{2x + 1} \, dx = \frac{1}{3} (\sqrt{2x + 1})^3 + C \)

### Initial Value Problems
65. Which of the following graphs shows the solution of the initial value problem
\[ \frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1? \]

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Give reasons for your answer.

66. Which of the following graphs shows the solution of the initial value problem
\[ \frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1? \]

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Give reasons for your answer.
Solve the initial value problems in Exercises 67–86.

67. \( \frac{dy}{dx} = 2x - 7, \quad y(2) = 0 \)
68. \( \frac{dy}{dx} = 10 - x, \quad y(0) = -1 \)
69. \( \frac{dy}{dx} = \frac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1 \)
70. \( \frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0 \)
71. \( \frac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5 \)
72. \( \frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0 \)
73. \( \frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4 \)
74. \( \frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1 \)
75. \( \frac{dr}{d\theta} = -\pi \sin \pi \theta, \quad r(0) = 0 \)
76. \( \frac{dr}{d\theta} = \cos \pi \theta, \quad r(0) = 1 \)
77. \( \frac{dv}{dt} = \frac{1}{2} \sec t \tan t, \quad v(0) = 1 \)
78. \( \frac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\frac{\pi}{2}\right) = -7 \)
79. \( \frac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1 \)
80. \( \frac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0 \)
81. \( \frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \frac{dr}{dt} \bigg|_{t=1} = 1, \quad r(1) = 1 \)
82. \( \frac{d^2s}{dt^2} = \frac{3t}{8}; \quad \frac{ds}{dt} \bigg|_{t=4} = 3, \quad s(4) = 4 \)
83. \( \frac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5 \)
84. \( \frac{d^2\theta}{dt^2} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\frac{1}{2}, \quad \theta(0) = \sqrt{2} \)
85. \( y^{(4)} = -\sin t + \cos t; \quad y''(0) = 7, \quad y'''(0) = y''(0) = -1, \quad y(0) = 0 \)
86. \( y^{(4)} = -\cos x + 8 \sin 2x; \quad y''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3 \)

**Finding Curves**

87. Find the curve \( y = f(x) \) in the \( xy \)-plane that passes through the point \((9, 4)\) and whose slope at each point is \( 3\sqrt{x} \).

88. a. Find a curve \( y = f(x) \) with the following properties:
   i) \( \frac{d^2y}{dx^2} = 6x \)
   ii) Its graph passes through the point \((0, 1)\), and has a horizontal tangent there.
   b. How many curves like this are there? How do you know?

**Solution (Integral) Curves**

Exercises 89–92 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

89. \( y \quad \frac{dy}{dx} = 1 - \frac{4}{3} x^{1/3} \)
90. \( y \quad \frac{dy}{dx} = x - 1 \)
91. \( y \quad \frac{dy}{dx} = \sin x - \cos x \)
92. \( y \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \pi \sin \pi x \)

**Applications**

93. **Finding displacement from an antiderivative of velocity**
   a. Suppose that the velocity of a body moving along the \( s \)-axis is
      \[ \frac{ds}{dt} = v = 9.8t - 3. \]
      i) Find the body’s displacement over the time interval from \( t = 1 \) to \( t = 3 \) given that \( s = 5 \) when \( t = 0 \).
      ii) Find the body’s displacement from \( t = 1 \) to \( t = 3 \) given that \( s = -2 \) when \( t = 0 \).
      iii) Now find the body’s displacement from \( t = 1 \) to \( t = 3 \) given that \( s = s_0 \) when \( t = 0 \).
96. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

97. Motion along a coordinate line A particle moves on a coordinate line with acceleration \( a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t}) \), subject to the conditions that \( ds/dt = 4 \) and \( s = 0 \) when \( t = 1 \). Find

- the velocity \( v = ds/dt \) in terms of \( t \)
- the position \( s \) in terms of \( t \)

98. The hammer and the feather When Apollo 15 astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for \( s \) as a function of \( t \). Then find the value of \( t \) that makes \( s \) equal to 0.

Differential equation: \( d^2s/dt^2 = -5.2 \) ft/sec²
Initial conditions: \( ds/dt = 0 \) and \( s = 4 \) when \( t = 0 \)

99. Motion with constant acceleration The standard equation for the position \( s \) of a body moving with a constant acceleration \( a \) along a coordinate line is

\[
s = \frac{a}{2} t^2 + v_0 t + s_0,
\]

where \( v_0 \) and \( s_0 \) are the body’s velocity and position at time \( t = 0 \). Derive this equation by solving the initial value problem

Differential equation: \( \frac{d^2s}{dt^2} = a \)

Initial conditions: \( \frac{ds}{dt} = v_0 \) and \( s = s_0 \) when \( t = 0 \).

100. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of \( g \) length-units/sec², Equation (1) in Exercise 99 takes the form

\[
s = -\frac{1}{2} g t^2 + v_0 t + s_0,
\]

where \( s \) is the body’s height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing \( s \). The velocity \( v_0 \) is positive if the object is rising at time \( t = 0 \) and negative if the object is falling.

Instead of using the result of Exercise 99, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

Theory and Examples

101. Suppose that

\[
f(x) = \frac{d}{dx}(1 - \sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).
\]

Find:

- \( \int f(x) \, dx \)
- \( \int g(x) \, dx \)
- \( \int [-f(x)] \, dx \)
- \( \int [-g(x)] \, dx \)
- \( \int [f(x) + g(x)] \, dx \)
- \( \int [f(x) - g(x)] \, dx \)

102. Uniqueness of solutions If differentiable functions \( y = F(x) \) and \( y = G(x) \) both solve the initial value problem

\[
\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,
\]

on an interval \( I \), must \( F(x) = G(x) \) for every \( x \) in \( I \)? Give reasons for your answer.
COMPUTER EXPLORATIONS

Use a CAS to solve the initial problems in Exercises 103–106. Plot the solution curves.

103. \( y' = \cos^2 x + \sin x, \ y(\pi) = 1 \)

104. \( y' = \frac{1}{x} + x, \ y(1) = -1 \)

105. \( y' = \frac{1}{\sqrt{4 - x^2}}, \ y(0) = 2 \)

106. \( y'' = \frac{2}{x} + \sqrt{x}, \ y(1) = 0, \ y'(1) = 0 \)