EXERCISES 7.3

Algebraic Calculations with the Exponential and Logarithm
Find simpler expressions for the quantities in Exercises 1–4.

\begin{align*}
1. \ a. \ e^{\ln 7.2} & \quad b. \ e^{-\ln x^2} & \quad c. \ e^{\ln x - \ln y} \\
2. \ a. \ e^{\ln (x + y)} & \quad b. \ e^{-\ln 0.3} & \quad c. \ e^{\ln x - \ln 2} \\
3. \ a. \ 2 \ln \sqrt{e} & \quad b. \ \ln (\ln e^x) & \quad c. \ \ln (e^{x^2 - y^2}) \\
4. \ a. \ \ln (e^{\sec \theta}) & \quad b. \ \ln (e^{\tan \theta}) & \quad c. \ \ln (e^{2 \ln x}) \\
\end{align*}

Solving Equations with Logarithmic or Exponential Terms
In Exercises 5–10, solve for $y$ in terms of $t$ or $x$, as appropriate.

\begin{align*}
5. \ \ln y &= 2t + 4 & \quad 6. \ \ln y &= -t + 5 \\
7. \ \ln (y - 40) &= 5t & \quad 8. \ \ln (1 - 2y) &= t \\
9. \ \ln (y - 1) - \ln 2 &= x + \ln x & \quad 10. \ \ln (y^2 - 1) - \ln (y + 1) &= \ln (\sin x) \\
\end{align*}

In Exercises 11 and 12, solve for $k$.

\begin{align*}
11. \ a. \ e^{2k} &= 4 & \quad b. \ 100e^{10k} &= 200 & \quad c. \ e^{k/1000} &= a \\
12. \ a. \ e^{\ln 4} &= \frac{1}{4} & \quad b. \ 80e^k &= 1 & \quad c. \ e^{(\ln 0.8)k} &= 0.8 \\
\end{align*}

In Exercises 13–16, solve for $t$.

\begin{align*}
13. \ a. \ e^{-0.3t} &= 27 & \quad b. \ e^{kt} &= \frac{1}{2} & \quad c. \ e^{(\ln 0.2)t} &= 0.4 \\
14. \ a. \ e^{-0.01t} &= 1000 & \quad b. \ e^{kt} &= \frac{1}{10} & \quad c. \ e^{(\ln 2)t} &= \frac{1}{2} \\
15. \ e^{\sqrt{t}} &= x^2 & \quad 16. \ e^{(x^2)e^{(2x+1)}} &= e^t \\
\end{align*}

Derivatives
In Exercises 17–36, find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

\begin{align*}
17. \ y &= e^{-5x} & \quad 18. \ y &= e^{2x/3} \\
19. \ y &= e^{5 - 7x} & \quad 20. \ y &= e^{(4\sqrt{x^2 + x})} \\
21. \ y &= xe^x - e^x & \quad 22. \ y &= (1 + 2x)e^{-2x} \\
23. \ y &= (x^2 - 2x + 2)e^x & \quad 24. \ y &= (9x^2 - 6x + 2)e^{3x} \\
25. \ y &= e^{(\sin \theta + \cos \theta)} & \quad 26. \ y &= \ln (3\theta e^{-\theta}) \\
27. \ y &= \cos (e^{-\theta}) & \quad 28. \ y &= \theta e^{-20} \cos 50 \\
29. \ y &= \ln (3e^{-\theta}) & \quad 30. \ y &= \ln (2e^{-\theta} \sin t) \\
31. \ y &= \ln \left(\frac{e^\theta}{1 + e^\theta}\right) & \quad 32. \ y &= \ln \left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right) \\
33. \ y &= e^{(x \cos t + \ln t)} & \quad 34. \ y &= e^{x^4/(\ln t^3 + 1)} \\
35. \ y &= \int_0^{\ln x} e^t \, dt & \quad 36. \ y &= \int_0^{e^{2\pi}} \ln t \, dt \\
\end{align*}

In Exercises 37–40, find $dy/dx$.

\begin{align*}
37. \ \ln y &= e^t \sin x & \quad 38. \ \ln xy^2 &= x^3 + x^2 \\
39. \ e^{2x} &= \sin (x + 3y) & \quad 40. \ \tan y &= e^t + \ln x \\
\end{align*}

Integrals
Evaluate the integrals in Exercises 41–62.

\begin{align*}
41. \ & \int (e^{3x} + 5e^{-x}) \, dx & \quad 42. \ & \int (2e^{x} - 3e^{-2x}) \, dx \\
43. \ & \int \ln^3 e^x \, dx & \quad 44. \ & \int_0^1 e^x \, dx \\
45. \ & \int 8e^{(x+1)} \, dx & \quad 46. \ & \int 2e^{(2x-1)} \, dx \\
47. \ & \int \ln^9 e^{x/2} \, dx & \quad 48. \ & \int_0^{\ln 16} e^{x^4} \, dx \\
49. \ & \int e^{\ln e^{\sqrt{7}}} \, dr & \quad 50. \ & \int \frac{e^{\ln e^{\sqrt{r}}}}{\sqrt{r}} \, dr \\
51. \ & \int 2t e^{-t^2} \, dt & \quad 52. \ & \int t^3 e^{(t^2)} \, dt \\
53. \ & \int \frac{e^{1/x}}{x^3} \, dx & \quad 54. \ & \int x^3 e^{-1/x} \, dx \\
55. \ & \int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta & \quad 56. \ & \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta \\
57. \ & \int e^{x^2} \sec x \, dx & \quad 58. \ & \int e^{\sec (\pi/2) \csc (\pi + t)} \cot (\pi + t) \, dt \\
\end{align*}
59. \[ \int_{\ln(\pi/2)}^{\ln(\pi/6)} 2e^u \cos e^u \, du \]
60. \[ \int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos (e^{x^2}) \, dx \]
61. \[ \int \frac{e^r}{1 + e^r} \, dr \]
62. \[ \int \frac{dx}{1 + e^x} \]

**Initial Value Problems**

Solve the initial value problems in Exercises 63–66.

63. \[ \frac{dy}{dt} = e^t \sin (e^t - 2), \quad y(0) = 0 \]
64. \[ \frac{dy}{dt} = e^{-t} \sec^2 (\pi e^{-t}), \quad y(0) = 2/\pi \]
65. \[ \frac{dy}{dx} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0 \]
66. \[ \frac{dy}{dx^2} = 1 - e^{2x}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0 \]

**Theory and Applications**

67. Find the absolute maximum and minimum values of \( f(x) = e^x - 2x \) on \([0, 1]\).

68. Where does the periodic function \( f(x) = 2e^{\sin(x/2)} \) take on its extreme values and what are these values?

69. Find the absolute maximum value of \( f(x) = x^2 \ln (1/x) \) and say where it is assumed.

70. Graph \( f(x) = (x - 3)^2e^x \) and its first derivative together. Comment on the behavior of \( f \) in relation to the signs and values of \( f' \). Identify significant points on the graphs with calculus, as necessary.

71. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve \( y = e^{x^2} \), below by the curve \( y = e^x \), and on the right by the line \( x = 3 \).

72. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve \( y = e^{x^2/2} \), below by the curve \( y = e^{-x^2/2} \), and on the right by the line \( x = 2 \ln 2 \).

73. Find a curve through the origin in the \( xy \)-plane whose length from \( x = 0 \) to \( x = 1 \) is

\[ L = \int_0^1 \sqrt{1 + \frac{1}{4} e^x} \, dx. \]

74. Find the area of the surface generated by revolving the curve \( x = (e^y + e^{-y})/2, \) \( 0 \leq y \leq \ln 2 \), about the \( y \)-axis.

75. a. Show that \( \int \ln x \, dx = x \ln x - x + C \).
   
   b. Find the average value of \( \ln x \) over \([1, e]\).

76. Find the average value of \( f(x) = 1/x \) on \([1, 2]\).

77. The linearization of \( e^x \) at \( x = 0 \)
   
   a. Derive the linear approximation \( e^x \approx 1 + x \) at \( x = 0 \).
   
   b. Estimate to five decimal places the magnitude of the error involved in replacing \( e^x \) by \( 1 + x \) on the interval \([0, 0.2]\).
   
   c. Graph \( e^x \) and \( 1 + x \) together for \(-2 \leq x \leq 2 \). Use different colors, if available. On what intervals does the approximation appear to overestimate \( e^x \)? Underestimate \( e^x \)?

78. Laws of Exponents
   
   a. Starting with the equation \( e^{x_1} e^{x_2} = e^{x_1 + x_2} \), derived in the text, show that \( e^{x^2} = 1/e^x \) for any real number \( x \). Then show that \( e^{x_1}/e^{x_2} = e^{x_1 - x_2} \) for any numbers \( x_1 \) and \( x_2 \).
   
   b. Show that \( (e^{x_1})^{x_2} = (e^{x_2})^{x_1} \) for any numbers \( x_1 \) and \( x_2 \).

79. A decimal representation of \( e \)
   
   Find \( e \) to as many decimal places as your calculator allows by solving the equation \( \ln x = 1 \).

80. The inverse relation between \( e^x \) and \( \ln x \)
   
   Find out how good your calculator is at evaluating the composites \( e^{\ln x} \) and \( \ln (e^x) \).

81. Show that for any number \( a > 1 \)

\[ \int_1^a \ln x \, dx + \int_0^{\ln a} e^y \, dy = a \ln a. \]

(See accompanying figure.)

82. The geometric, logarithmic, and arithmetic mean inequality
   
   a. Show that the graph of \( e^x \) is concave up over every interval of \( x \)-values.
b. Show, by reference to the accompanying figure, that if $0 < a < b$ then
\[ e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a). \]

c. Use the inequality in part (b) to conclude that
\[ \sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}. \]

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.