EXERCISES 7.4

Algebraic Calculations With \( a^x \) and \( \log_a x \)

Simplify the expressions in Exercises 1–4.

1. a. \( 5^{\log_7} \)  b. \( 8^{\log_2} \)  c. \( 1.3^{\log_{1.75}} \)
2. a. \( 2^{\log_3} \)  b. \( 10^{\log_{10}(1/2)} \)  c. \( \pi^{\log_2} \)
3. a. \( \log_{121} \)  b. \( \log_{121} 11 \)  c. \( \log_3 \left( \frac{1}{3} \right) \)
4. a. \( 2^{5\log_2(x^2)} \)  b. \( \log_2(e^{\ln 2}) \)  c. \( \log_4(2^{e^{\sin x}}) \)

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. a. \( \frac{\log_2 x}{\log_3 x} \)  b. \( \frac{\log_3 x}{\log_5 x} \)  c. \( \frac{\log x}{\log a} \)
6. a. \( \frac{\log_4 x}{\log_{\sqrt{2}} x} \)  b. \( \frac{\log_{\sqrt{2}} x}{\log_4 x} \)  c. \( \frac{\log b}{\log a} \)

Solve the equations in Exercises 7–10 for \( x \).

7. \( 3^{\log_3(7)} + 2^{\log_2(5)} = 5^{\log_5(2)} \)
8. \( 8^{\log_3(5)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)} \)
9. \( 3^{\log_3(x^2)} = 5^{\ln x} - 3 \cdot 10^{\log_{10}(2)} \)
10. \( \ln e + 4^{-2\log_e(x)} = \frac{1}{x} \log_{10}(100) \)

Derivatives

In Exercises 11–38, find the derivative of \( y \) with respect to the given independent variable.

11. \( y = 2^x \)  12. \( y = 3^{-x} \)
13. \( y = 5^{\sqrt{x}} \)  14. \( y = 2^{(x^2)} \)
15. \( y = x^\pi \)  16. \( y = 1^x - e \)
17. \( y = (\cos \theta)^{\sqrt{2}} \)  18. \( y = (\ln \theta)\theta \)
19. \( y = \sec \theta \ln 7 \)  20. \( y = 3\tan \theta \ln 3 \)
21. \( y = 2^{\sin \theta} \)  22. \( y = 5^{\cos \theta} \)
23. \( y = \log_2 5\theta \)  24. \( y = \log_3(1 + \theta \ln 3) \)
25. \( y = \log_4 x + \log_4 x^2 \)  26. \( y = \log_2 x^3 - \log_5 \sqrt{x} \)
27. \( y = \log_2 r \cdot \log_4 r \)  28. \( y = \log_1 r \cdot \log_9 r \)
29. \( y = \log_3 \left( \frac{x + 1}{x - 1} \right) \ln^3 \)  30. \( y = \log_5 \left( \frac{7x}{3x + 2} \right)^{\ln 3} \)
31. \( y = \theta \sin (\log_7 \theta) \)  32. \( y = \log_8 \left( \frac{\sin \theta \cos \theta}{\cos^2 \theta} \right) \)
33. \( y = \log_9 e^x \)  34. \( y = \log_6 \left( \frac{x^2^e}{2\sqrt{x} + 1} \right) \)
35. \( y = 3^{\log_2 t} \)  36. \( y = 3 \log_8 (\log_2 t) \)
37. \( y = \log_2 (2 \ln 2) \)  38. \( y = \log_3 (e^{(3\sin t)/(\ln 3)}) \)

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of \( y \) with respect to the given independent variable.

39. \( y = (x + 1)^x \)  40. \( y = x^{(x+1)} \)
41. \( y = \sqrt[3]{x} \)  42. \( y = x^{3/2} \)
43. \( y = (\sin x)^x \)  44. \( y = x^{\sin x} \)
45. \( y = x^{\ln x} \)  46. \( y = (\ln x)^{\ln x} \)

Integration

Evaluate the integrals in Exercises 47–56.

47. \( \int 5^x \, dx \)  48. \( \int (1.3)^x \, dx \)
49. \( \int_0^1 2^{-\theta} \, d\theta \)  
50. \( \int_0^S 5^{-\theta} \, d\theta \)

51. \( \int_1^\sqrt{2} x^2 \, dx \)  
52. \( \int_1^\frac{\pi}{2} \frac{x^2}{\sqrt{x}} \, dx \)

53. \( \int_0^{\pi/2} \sin t \, dt \)  
54. \( \int_0^{\pi/4} \left( \frac{1}{3} \right)^{\tan t} \sec^2 t \, dt \)

55. \( \int_2^4 x^3(1 + \ln x) \, dx \)  
56. \( \int_2^4 \frac{2\ln x}{x} \, dx \)

Evaluate the integrals in Exercises 57–60.

57. \( \int 3x^{\sqrt{3}} \, dx \)

58. \( \int x^{\sqrt{2}-1} \, dx \)

59. \( \int_0^3 (\sqrt{2} + 1)x^{\sqrt{2}} \, dx \)

60. \( \int_1^{e^{(\ln 2)-1}} \frac{e^x}{x} \, dx \)

Evaluate the integrals in Exercises 61–70.

61. \( \int \frac{\log_{10} x}{x} \, dx \)

62. \( \int \frac{4 \log_2 x}{x} \, dx \)

63. \( \int_1^4 \frac{2 \log_2 x}{x} \, dx \)

64. \( \int_1^4 \frac{2 \ln 10 \log_{10} x}{x} \, dx \)

65. \( \int_0^2 \frac{\log_2 (x + 2)}{x + 2} \, dx \)

66. \( \int_0^2 \frac{\log_{10} (10x)}{x} \, dx \)

67. \( \int_3^2 \frac{\log_10 (x + 1)}{x + 1} \, dx \)

68. \( \int_2^3 \frac{2 \log_2 (x - 1)}{x - 1} \, dx \)

69. \( \int \frac{dx}{x \log_{10} x} \)

70. \( \int \frac{dx}{x (\log_{10} x)^2} \)

Evaluate the integrals in Exercises 71–74.

71. \( \int_1^{\ln x} \frac{1}{t} \, dt, \quad x > 1 \)

72. \( \int_1^{e^x} \frac{1}{t} \, dt \)

73. \( \int_1^{\ln a} \frac{1}{t} \, dt, \quad x > 0 \)

74. \( \frac{1}{\ln a} \int_1^{\ln a} \frac{1}{t} \, dt, \quad x > 0 \)

Theory and Applications

75. Find the area of the region between the curve \( y = 2x/(1 + x^2) \) and the interval \(-2 \leq x \leq 2\) of the x-axis.

76. Find the area of the region between the curve \( y = 2^{1-x} \) and the interval \(-1 \leq x \leq 1\) of the x-axis.

77. Blood pH The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for \([H_3O^+]\).

78. Brain fluid pH The cerebrospinal fluid in the brain has a hydronium ion concentration of about \([H_3O^+] = 4.8 \times 10^{-8}\) moles per liter. What is the pH?

79. Audio amplifiers By what factor \( k \) do you have to multiply the intensity of \( f(t) \) of the sound from your audio amplifier to add 10 dB to the sound level?

80. Audio amplifiers You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration \([H_3O^+]\) (moles/L) and the hydroxyl ion concentration \([OH^-]\) (moles/L) is about \(10^{-14}\).
   a. What value of \([H_3O^+]\) minimizes the sum of the concentrations, \( S = [H_3O^+] + [OH^-] \)? (Hint: Change notation. Let \( x = [H_3O^+]\).)
   b. What is the pH of a solution in which \( S \) has this minimum value?
   c. What ratio of \([H_3O^+]\) to \([OH^-]\) minimizes \( S \)?

82. Could \( \log_a b \) possibly equal \( 1/\log_b a \)? Give reasons for your answer.

83. The equation \( x^2 = 2^x \) has three solutions: \( x = 2 \), \( x = 4 \), and one other. Estimate the third solution as accurately as you can by graphing.

84. Could \( x^{\ln x} \) possibly be the same as \( e^{\ln x} \) for \( x > 0 \)? Graph the two functions and explain what you see.

85. The linearization of \( 2^x \)
   a. Find the linearization of \( f(x) = 2^x \) at \( x = 0 \). Then round its coefficients to two decimal places.
   b. Graph the linearization and function together for \(-3 \leq x \leq 3 \) and \(-1 \leq x \leq 1\).

86. The linearization of \( \log_3 x \)
   a. Find the linearization of \( f(x) = \log_3 x \) at \( x = 3 \). Then round its coefficients to two decimal places.
   b. Graph the linearization and function together in the window \( 0 \leq x \leq 8 \) and \( 2 \leq x \leq 4 \).

Calculations with Other Bases

87. Most scientific calculators have keys for \( \log_{10} x \) and \( \ln x \). To find logarithms to other bases, we use the Equation (5), \( \log_b x = (\ln x)/(\ln a) \).
   Find the following logarithms to five decimal places.
   a. \( \log_3 8 \)
   b. \( \log_7 0.5 \)
   c. \( \log_{20} 17 \)
   d. \( \log_{0.5} 7 \)
   e. \( \ln x \), given that \( \log_{10} x = 2.3 \)
   f. \( \ln x \), given that \( \log_2 x = 1.4 \)
   g. \( \ln x \), given that \( \log_{10} x = -1.5 \)
   h. \( \ln x \), given that \( \log_{10} x = -0.7 \)

88. Conversion factors
   a. Show that the equation for converting base 10 logarithms to base 2 logarithms is \( \log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x \).
   b. Show that the equation for converting base \( a \) logarithms to base \( b \) logarithms is \( \log_b x = \frac{\ln a}{\ln b} \log_a x \).
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89. Orthogonal families of curves  Prove that all curves in the family

\[ y = -\frac{1}{2} x^2 + k \]

(k any constant) are perpendicular to all curves in the family

\[ y = \ln x + c \] (c any constant) at their points of intersection. (See the accompanying figure.)

| 90. The inverse relation between \( e^x \) and \( \ln x \) | Find out how good your calculator is at evaluating the composites

\[ e^{\ln x} \] and \( \ln(e^x) \).

| 91. A decimal representation of \( e \) | Find \( e \) to as many decimal places as your calculator allows by solving the equation \( \ln x = 1 \).

| 92. Which is bigger, \( \pi^e \) or \( e^{\pi} \)? | Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though.

a. Find an equation for the line through the origin tangent to the graph of \( y = \ln x \).

b. Give an argument based on the graphs of \( y = \ln x \) and the tangent line to explain why \( \ln x < x/e \) for all positive \( x \neq e \).

c. Show that \( \ln(x^e) < x \) for all positive \( x \neq e \).

d. Conclude that \( x^e < e^x \) for all positive \( x \neq e \).

e. So which is bigger, \( \pi^e \) or \( e^\pi \)?