EXERCISES 7.6

Comparisons with the Exponential $e^x$

1. Which of the following functions grow faster than $e^x$ as $x \to \infty$? Which grow at the same rate as $e^x$? Which grow slower?
   a. $x + 3$
   b. $x^3 + \sin^2 x$
   c. $\sqrt{x}$
   d. $4^x$
   e. $(3/2)^x$
   f. $e^{\sqrt{x}}$
   g. $e^{x/2}$
   h. $\log_{10} x$

2. Which of the following functions grow faster than $e^x$ as $x \to \infty$? Which grow at the same rate as $e^x$? Which grow slower?
   a. $10x^4 + 30x + 1$
   b. $x \ln x - x$
   c. $\sqrt{1 + x^2}$
   d. $(5/2)^x$
   e. $e^{-x}$
   f. $xe^x$
   g. $e^{\cos x}$
   h. $e^{x^{-1}}$

Comparisons with the Power $x^2$

3. Which of the following functions grow faster than $x^2$ as $x \to \infty$? Which grow at the same rate as $x^2$? Which grow slower?
   a. $x^2 + 4x$
   b. $x^5 - x^2$
   c. $\sqrt{x^4 + x^3}$
   d. $(x + 3)^2$
   e. $x \ln x$
   f. $2^x$
   g. $x^3 e^{-x}$
   h. $8x^2$

4. Which of the following functions grow faster than $x^2$ as $x \to \infty$? Which grow at the same rate as $x^2$? Which grow slower?
   a. $x^2 + \sqrt{x}$
   b. $10x^2$
   c. $x^2 e^{-x}$
   d. $\log_{10} (x^2)$
   e. $x^3 - x^2$
   f. $(1/10)^y$
   g. $(1.1)^y$
   h. $x^2 + 100x$
Comparisons with the Logarithm \( \ln x \)

5. Which of the following functions grow faster than \( \ln x \) as \( x \to \infty \)? Which grow at the same rate as \( \ln x \)? Which grow slower?
   a. \( \log_3 x \)
   b. \( \log_{10} 2x \)
   c. \( \ln \sqrt{x} \)
   d. \( \sqrt{x} \)
   e. \( x \)
   f. \( 5 \ln x \)
   g. \( \frac{1}{x} \)
   h. \( e^x \)

6. Which of the following functions grow faster than \( \ln x \) as \( x \to \infty \)? Which grow at the same rate as \( \ln x \)? Which grow slower?
   a. \( \log_2 \left( x^2 \right) \)
   b. \( \log_{10} \left( 10x \right) \)
   c. \( \frac{1}{\sqrt{x}} \)
   d. \( \frac{1}{x^2} \)
   e. \( x - 2 \ln x \)
   f. \( e^{-x} \)
   g. \( \ln (\ln x) \)
   h. \( \ln \left( 2x + 5 \right) \)

Ordering Functions by Growth Rates

7. Order the following functions from slowest growing to fastest growing as \( x \to \infty \).
   a. \( e^x \)
   b. \( x^2 \)
   c. \( (\ln x)^x \)
   d. \( e^{\sqrt{x}} \)

8. Order the following functions from slowest growing to fastest growing as \( x \to \infty \).
   a. \( 2^x \)
   b. \( x^2 \)
   c. \( (\ln 2)^x \)
   d. \( e^x \)

Big-oh and Little-oh; Order

9. True, or false? As \( x \to \infty \),
   a. \( x = o(x) \)
   b. \( x = o(x + 5) \)
   c. \( x = O(x + 5) \)
   d. \( x = O(2x) \)
   e. \( e^x = o(e^{2x}) \)
   f. \( x + \ln x = O(x) \)
   g. \( \ln x = o(\ln 2x) \)
   h. \( \sqrt{x^2 + 5} = O(x) \)

10. True, or false? As \( x \to \infty \),
    a. \( \frac{1}{x + 3} = O\left( \frac{1}{x} \right) \)
    b. \( \frac{1}{x} + \frac{1}{x^2} = O\left( \frac{1}{x^2} \right) \)
    c. \( \frac{1}{x} - \frac{1}{x^2} = o\left( \frac{1}{x^2} \right) \)
    d. \( 2 + \cos x = O(2) \)
    e. \( e^x + x = O(e^x) \)
    f. \( x \ln x = o(x^2) \)
    g. \( \ln (\ln x) = O(\ln x) \)
    h. \( \ln (x) = o(\ln (x^2 + 1)) \)

11. Show that if positive functions \( f(x) \) and \( g(x) \) grow at the same rate as \( x \to \infty \), then \( f = O(g) \) and \( g = O(f) \).
12. When is a polynomial \( f(x) \) of smaller order than a polynomial \( g(x) \) as \( x \to \infty \)? Give reasons for your answer.
13. When is a polynomial \( f(x) \) of at most the order of a polynomial \( g(x) \) as \( x \to \infty \)? Give reasons for your answer.

14. What do the conclusions we drew in Section 2.4 about the limits of rational functions tell us about the relative growth of polynomials as \( x \to \infty \)?

Other Comparisons

15. Investigate

\[ \lim_{x \to \infty} \frac{\ln (x + 1)}{\ln x} \quad \text{and} \quad \lim_{x \to \infty} \frac{\ln (x + 999)}{\ln x}. \]

Then use l'Hôpital's Rule to explain what you find.

16. (Continuation of Exercise 15.) Show that the value of

\[ \lim_{x \to \infty} \frac{\ln (x + a)}{\ln x} \]

is the same no matter what value you assign to the constant \( a \). What does this say about the relative rates at which the functions \( f(x) = \ln (x + a) \) and \( g(x) = \ln x \) grow?

17. Show that \( \sqrt{10x + 1} \) and \( \sqrt{x + 1} \) grow at the same rate as \( x \to \infty \) by showing that they both grow at the same rate as \( \sqrt{x} \) as \( x \to \infty \).

18. Show that \( \sqrt{x^2 + x} \) and \( \sqrt{x^2 - x^2} \) grow at the same rate as \( x \to \infty \) by showing that they both grow at the same rate as \( x^2 \) as \( x \to \infty \).

19. Show that \( e^x \) grows faster as \( x \to \infty \) than \( x^n \) for any positive integer \( n \), even \( x^{\frac{1,000,000}{1}} \) \( (Hint: \) What is the \( n \)th derivative of \( e^x \)?

20. The function \( e^x \) outgrows any polynomial \( \) Show that \( e^x \) grows faster as \( x \to \infty \) than any polynomial

\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \]

21. a. Show that \( \ln x \) grows slower as \( x \to \infty \) than \( x^{1/n} \) for any positive integer \( n \), even \( x^{1/1,000,000} \).

b. Although the values of \( x^{1/1,000,000} \) eventually overtake the values of \( \ln x \), you have to go way out on the \( x \)-axis before this happens. Find a value of \( x \) greater than 1 for which \( x^{1/1,000,000} > \ln x \). You might start by observing that when \( x > 1 \) the equation \( \ln x = x^{1/1,000,000} \) is equivalent to the equation \( \ln (\ln x) = (\ln x)/1,000,000 \).

c. Even \( x^{1/10} \) takes a long time to overtake \( \ln x \). Experiment with a calculator to find the value of \( x \) at which the graphs of \( x^{1/10} \) and \( \ln x \) cross, or, equivalently, at which \( x = 10 \ln (\ln x) \). Bracket the crossing point between powers of 10 and then close in by successive halving.

d. (Continuation of part (c).) The value of \( x \) at which \( \ln x = 10 \ln (\ln x) \) is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. The function \( \ln x \) grows slower than any polynomial \( \) Show that \( \ln x \) grows slower as \( x \to \infty \) than any nonconstant polynomial.
**Algorithms and Searches**

23. a. Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

\[ n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2. \]

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

b. Graph the functions in part (a) together to get a sense of how rapidly each one grows.

24. Repeat Exercise 23 for the functions

\[ n, \quad \sqrt{n \log_2 n}, \quad (\log_2 n)^2. \]

25. Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? A binary search?

26. You are looking for an item in an ordered list 450,000 items long (the length of *Webster’s Third New International Dictionary*). How many steps might it take to find the item with a sequential search? A binary search?