7.7 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However we can restrict their domains to intervals on which they are one-to-one. The sine function increases from −1 at $x = -\pi/2$ to +1 at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 7.16). Similar domain restrictions can be applied to all six trigonometric functions.

Domain restrictions that make the trigonometric functions one-to-one

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$[-\pi/2, \pi/2]$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$[0, \pi]$</td>
<td>$[-1, 1]$</td>
</tr>
</tbody>
</table>

FIGURE 7.16 The graph of $y = \sin^{-1} x$. 
Chapter 7: Transcendental Functions

Since these restricted functions are now one-to-one, they have inverses, which we denote by

\[ y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x \]
\[ y = \cos^{-1} x \quad \text{or} \quad y = \arccos x \]
\[ y = \tan^{-1} x \quad \text{or} \quad y = \arctan x \]
\[ y = \cot^{-1} x \quad \text{or} \quad y = \arccot x \]
\[ y = \sec^{-1} x \quad \text{or} \quad y = \text{arcsec } x \]
\[ y = \csc^{-1} x \quad \text{or} \quad y = \text{arccsc } x \]

These equations are read “y equals the arcsine of x” or “y equals arcsin x” and so on.
CAUTION  The $-1$ in the expressions for the inverse means “inverse.” It does not mean reciprocal. For example, the reciprocal of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 7.17. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$, as in Section 7.1. We now take a closer look at these functions and their derivatives.

**FIGURE 7.17**  Graphs of the six basic inverse trigonometric functions.

### The Arcsine and Arccosine Functions

The arcsine of $x$ is the angle in $[-\pi/2, \pi/2]$ whose sine is $x$. The arccosine is an angle in $[0, \pi]$ whose cosine is $x$. 

---

*Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley*
### DEFINITION Arcsine and Arccosine Functions

\[ y = \sin^{-1} x \] is the number in \([-\pi/2, \pi/2]\) for which \(\sin y = x\).

\[ y = \cos^{-1} x \] is the number in \([0, \pi]\) for which \(\cos y = x\).

#### The “Arc” in Arc Sine and Arc Cosine

The accompanying figure gives a geometric interpretation of \(y = \sin^{-1} x\) and \(y = \cos^{-1} x\) for radian angles in the first quadrant. For a unit circle, the equation \(s = r\theta\) becomes \(s = \theta\), so central angles and the arcs they subtend have the same measure. If \(x = \sin y\), then, in addition to being the angle whose sine is \(x\), \(y\) is also the length of arc on the unit circle that subtends an angle whose sine is \(x\). So we call \(y\) “the arc whose sine is \(x\).”

Known values of \(\sin x\) and \(\cos x\) can be inverted to find values of \(\sin^{-1} x\) and \(\cos^{-1} x\).
7.7 Inverse Trigonometric Functions

### Example 1: Common Values of $\sin^{-1} x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sin^{-1} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{3}/2$</td>
<td>$\pi/3$</td>
</tr>
<tr>
<td>$\sqrt{2}/2$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$\pi/6$</td>
</tr>
<tr>
<td>$-1/2$</td>
<td>$-\pi/6$</td>
</tr>
<tr>
<td>$-\sqrt{2}/2$</td>
<td>$-\pi/4$</td>
</tr>
<tr>
<td>$-\sqrt{3}/2$</td>
<td>$-\pi/3$</td>
</tr>
</tbody>
</table>

The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.

### Example 2: Common Values of $\cos^{-1} x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\cos^{-1} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{3}/2$</td>
<td>$\pi/6$</td>
</tr>
<tr>
<td>$\sqrt{2}/2$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$\pi/3$</td>
</tr>
<tr>
<td>$-1/2$</td>
<td>$2\pi/3$</td>
</tr>
<tr>
<td>$-\sqrt{2}/2$</td>
<td>$3\pi/4$</td>
</tr>
<tr>
<td>$-\sqrt{3}/2$</td>
<td>$5\pi/6$</td>
</tr>
</tbody>
</table>

The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$.

### Identities Involving Arcsine and Arccosine

As we can see from Figure 7.20, the arccosine of $x$ satisfies the identity

$$\cos^{-1} x + \cos^{-1} (-x) = \pi,$$

or

$$\cos^{-1} (-x) = \pi - \cos^{-1} x.$$

Also, we can see from the triangle in Figure 7.21 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2.$$

Equation (4) holds for the other values of $x$ in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 7.21. It is, however, a consequence of Equations (1) and (3) (Exercise 131).

### Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The arctangent of $x$ is an angle whose tangent is $x$. The arccotangent of $x$ is an angle whose cotangent is $x$. 

---

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley
We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of \( y = \tan^{-1} x \) is the number in \((-\pi/2, \pi/2)\) for which \( \tan y = x \).

The graph of \( y = \cot^{-1} x \) is the number in \((0, \pi)\) for which \( \cot y = x \).

The inverses of the restricted forms of \( \sec x \) and \( \csc x \) are chosen to be the functions graphed in Figures 7.24 and 7.25.

\[ \sec^{-1} x \quad \text{and} \quad \csc^{-1} x \]

\[ y = \tan^{-1} x \quad \text{Domain: } (-\infty, \infty) \quad \text{Range: } (-\pi/2, \pi/2) \]

\[ y = \cot^{-1} x \quad \text{Domain: } (-\infty, \infty) \quad \text{Range: } (0, \pi) \]

\[ y = \sec^{-1} x \quad \text{Domain: } |x| \geq 1 \quad \text{Range: } [0, \pi/2) \cup (\pi/2, \pi] \]

\[ y = \csc^{-1} x \quad \text{Domain: } |x| \geq 1 \quad \text{Range: } [-\pi/2, 0) \cup (0, \pi/2] \]

\[ \sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right) = \frac{\pi}{2} - \sin^{-1} \left( \frac{1}{x} \right) \quad (5) \]

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of \( y = \tan^{-1} x \) is symmetric about the origin because it is a branch of the graph \( x = \tan y \) that is symmetric about the origin (Figure 7.22). Algebraically this means that

\[ \tan^{-1} (-x) = -\tan^{-1} x; \]

the arctangent is an odd function. The graph of \( y = \cot^{-1} x \) has no such symmetry (Figure 7.23).

The inverses of the restricted forms of \( \sec x \) and \( \csc x \) are chosen to be the functions graphed in Figures 7.24 and 7.25.

**CAUTION** There is no general agreement about how to define \( \sec^{-1} x \) for negative values of \( x \). We chose angles in the second quadrant between \( \pi/2 \) and \( \pi \). This choice makes \( \sec^{-1} x \) an increasing function on each interval of its domain. Some tables choose \( \sec^{-1} x \) to lie in \([-\pi, -\pi/2)\) for \( x < 0 \) and some texts choose it to lie in \([\pi, 3\pi/2)\) (Figure 7.26). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation \( \sec^{-1} x = \cos^{-1} (1/x) \). From this, we can derive the identity

\[ \sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right) = \frac{\pi}{2} - \sin^{-1} \left( \frac{1}{x} \right) \]

by applying Equation (4).
7.7 Inverse Trigonometric Functions

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\tan^{-1}x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{3}$</td>
<td>$\pi/3$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>$\sqrt{3}/3$</td>
<td>$\pi/6$</td>
</tr>
<tr>
<td>$-\sqrt{3}/3$</td>
<td>$-\pi/6$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-\pi/4$</td>
</tr>
<tr>
<td>$-\sqrt{3}$</td>
<td>$-\pi/3$</td>
</tr>
</tbody>
</table>

The angles come from the first and fourth quadrants because the range of $\tan^{-1}x$ is $(-\pi/2, \pi/2)$.

**EXAMPLE 4** Find $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, $\csc \alpha$, and $\cot \alpha$ if

$$\alpha = \sin^{-1} \frac{2}{3}.$$  

**Solution** This equation says that $\sin \alpha = 2/3$. We picture $\alpha$ as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Figure 7.27). The length of the remaining side is

$$\sqrt{(3)^2 - (2)^2} = \sqrt{9 - 4} = \sqrt{5}.$$  

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos \alpha = \frac{\sqrt{5}}{3}, \quad \tan \alpha = \frac{2}{\sqrt{5}}, \quad \sec \alpha = \frac{3}{\sqrt{5}}, \quad \csc \alpha = \frac{3}{2}, \quad \cot \alpha = \frac{\sqrt{5}}{2}.$$  

**EXAMPLE 5** Find $\sec \left( \tan^{-1} \frac{x}{3} \right)$.

**Solution** We let $\theta = \tan^{-1} \left( \frac{x}{3} \right)$ (to give the angle a name) and picture $\theta$ in a right triangle with

$$\tan \theta = \frac{x}{3}.$$  

The length of the triangle’s hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$  

Thus,

$$\sec \left( \tan^{-1} \frac{x}{3} \right) = \sec \theta = \frac{\sqrt{x^2 + 9}}{3}.$$  

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley
EXAMPLE 6 Drift Correction

During an airplane flight from Chicago to St. Louis the navigator determines that the plane is 12 mi off course, as shown in Figure 7.28. Find the angle \( a \) for a course parallel to the original, correct course, the angle \( b \), and the correction angle \( c = a + b \).

Solution

\[
\begin{align*}
a &= \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8^\circ \\
b &= \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2^\circ \\
c &= a + b \approx 15^\circ.
\end{align*}
\]

The Derivative of \( y = \sin^{-1} u \)

We know that the function \( x = \sin y \) is differentiable in the interval \(-\pi/2 < y < \pi/2\) and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function \( y = \sin^{-1} x \) is differentiable throughout the interval \(-1 < x < 1\). We cannot expect it to be differentiable at \( x = 1 \) or \( x = -1 \) because the tangents to the graph are vertical at these points (see Figure 7.29).

We find the derivative of \( y = \sin^{-1} x \) by applying Theorem 1 with \( f(x) = \sin x \) and \( f^{-1}(x) = \sin^{-1} x \).

\[
\begin{align*}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} & \text{Theorem 1} \\
&= \frac{1}{\cos (\sin^{-1} x)} & f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2 (\sin^{-1} x)}} & \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}} & \sin (\sin^{-1} x) = x
\end{align*}
\]

Alternate Derivation: Instead of applying Theorem 1 directly, we can find the derivative of \( y = \sin^{-1} x \) using implicit differentiation as follows:

\[
\begin{align*}
\sin y &= x & y = \sin^{-1} x \Leftrightarrow \sin y = x \\
\frac{d}{dx} (\sin y) &= 1 & \text{Derivative of both sides with respect to } x \\
\cos y \frac{dy}{dx} &= 1 & \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\cos y} \quad \text{We can divide because } \cos y > 0 \\
&= \frac{1}{\sqrt{1 - x^2}} & \cos y = \sqrt{1 - \sin^2 y}
\end{align*}
\]
No matter which derivation we use, we have that the derivative of \( y = \sin^{-1} x \) with respect to \( x \) is

\[
\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.
\]

If \( u \) is a differentiable function of \( x \) with \( |u| < 1 \), we apply the Chain Rule to get

\[
\frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.
\]

**EXAMPLE 7** Applying the Derivative Formula

\[
\frac{d}{dx} (\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx} (x^2) = \frac{2x}{\sqrt{1 - x^4}}.
\]

*The Derivative of \( y = \tan^{-1} u \)*

We find the derivative of \( y = \tan^{-1} x \) by applying Theorem 1 with \( f(x) = \tan x \) and \( f^{-1}(x) = \tan^{-1} x \). Theorem 1 can be applied because the derivative of \( \tan x \) is positive for \(-\pi/2 < x < \pi/2\).

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x)))} \quad \text{Theorem 1}
\]

\[
= \frac{1}{\sec^2 (\tan^{-1} x)} = \frac{1}{1 + \tan^2 (\tan^{-1} x)} = \frac{1}{1 + x^2}
\]

The derivative is defined for all real numbers. If \( u \) is a differentiable function of \( x \), we get the Chain Rule form:

\[
\frac{d}{dx} (\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.
\]

**EXAMPLE 8** A Moving Particle

A particle moves along the \( x \)-axis so that its position at any time \( t \geq 0 \) is \( x(t) = \tan^{-1} \sqrt{t} \). What is the velocity of the particle when \( t = 16 \)?

*Solution*

\[
v(t) = \frac{d}{dt} \tan^{-1} \sqrt{t} = \frac{1}{1 + (\sqrt{t})^2} \cdot \frac{d}{dt} \sqrt{t} = \frac{1}{1 + t} \cdot \frac{1}{2\sqrt{t}}
\]
When \( t = 16 \), the velocity is
\[
v(16) = \frac{1}{1 + 16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}.
\]

**The Derivative of \( y = \sec^{-1} u \)**

Since the derivative of \( \sec x \) is positive for \( 0 < x < \pi/2 \) and \( \pi/2 < x < \pi \), Theorem 1 says that the inverse function \( y = \sec^{-1} x \) is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of \( y = \sec^{-1} x \), \( |x| > 1 \), using implicit differentiation and the Chain Rule as follows:

\[
y = \sec^{-1} x
\]
\[
\sec y = x
\]
Differentiate both sides.

\[
\frac{d}{dx} (\sec y) = \frac{d}{dx} x
\]
Inverse function relationship

\[
\sec y \tan y \frac{dy}{dx} = 1
\]
Chain Rule

\[
\frac{dy}{dx} = \frac{1}{\sec y \tan y}
\]
Since \( |x| > 1 \), \( y \) lies in \( (0, \pi/2) \cup (\pi/2, \pi) \) and \( \sec y \tan y \neq 0 \).

To express the result in terms of \( x \), we use the relationships
\[
\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}
\]
to get
\[
\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.
\]

Can we do anything about the \( \pm \) sign? A glance at Figure 7.30 shows that the slope of the graph \( y = \sec^{-1} x \) is always positive. Thus,
\[
\frac{d}{dx} \sec^{-1} x = \begin{cases} 
  + \frac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\
  - \frac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1.
\end{cases}
\]

With the absolute value symbol, we can write a single expression that eliminates the “\( \pm \)” ambiguity:
\[
\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.
\]

If \( u \) is a differentiable function of \( x \) with \( |u| > 1 \), we have the formula
\[
\frac{d}{dx} (\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.
\]
EXAMPLE 9 Using the Formula

\[ \frac{d}{dx} \sec^{-1} (5x^4) = \frac{1}{|5x^4\sqrt{(5x^4)^2 - 1}|} \frac{d}{dx} (5x^4) \]
\[ = \frac{1}{5x^4\sqrt{25x^8 - 1}} (20x^3) \quad 5x^4 > 0 \]
\[ = \frac{4}{x\sqrt{25x^8 - 1}} \]

Derivatives of the Other Three
We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is a much easier way, thanks to the following identities.

<table>
<thead>
<tr>
<th>Inverse Function–Inverse Cofunction Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos^{-1} x = \pi/2 - \sin^{-1} x )</td>
</tr>
<tr>
<td>( \cot^{-1} x = \pi/2 - \tan^{-1} x )</td>
</tr>
<tr>
<td>( \csc^{-1} x = \pi/2 - \sec^{-1} x )</td>
</tr>
</tbody>
</table>

We saw the first of these identities in Equation (4). The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of \( \cos^{-1} x \) is calculated as follows:

\[ \frac{d}{dx} (\cos^{-1} x) = \frac{d}{dx} \left( \frac{\pi}{2} - \sin^{-1} x \right) \quad \text{Identity} \]
\[ = -\frac{d}{dx} (\sin^{-1} x) \]
\[ = -\frac{1}{\sqrt{1 - x^2}} \quad \text{Derivative of arcsine} \]

EXAMPLE 10 A Tangent Line to the Arccotangent Curve
Find an equation for the line tangent to the graph of \( y = \cot^{-1} x \) at \( x = -1 \).

Solution First we note that
\[ \cot^{-1} (-1) = \pi/2 - \tan^{-1} (-1) = \pi/2 - (-\pi/4) = 3\pi/4. \]
The slope of the tangent line is
\[ \frac{dy}{dx} \bigg|_{x=-1} = \frac{1}{1 + x^2} \bigg|_{x=-1} = -\frac{1}{1 + (-1)^2} = -\frac{1}{2}, \]
so the tangent line has equation \( y - 3\pi/4 = (-1/2)(x + 1) \).
The derivatives of the inverse trigonometric functions are summarized in Table 7.3.

### TABLE 7.3 Derivatives of the inverse trigonometric functions

<table>
<thead>
<tr>
<th></th>
<th>Derivative Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \frac{d}{dx}(\sin^{-1} u) = \frac{du}{\sqrt{1 - u^2}} ), (</td>
</tr>
<tr>
<td>2.</td>
<td>( \frac{d}{dx}(\cos^{-1} u) = -\frac{du}{\sqrt{1 - u^2}} ), (</td>
</tr>
<tr>
<td>3.</td>
<td>( \frac{d}{dx}(\tan^{-1} u) = \frac{du}{1 + u^2} )</td>
</tr>
<tr>
<td>4.</td>
<td>( \frac{d}{dx}(\cot^{-1} u) = -\frac{du}{1 + u^2} )</td>
</tr>
<tr>
<td>5.</td>
<td>( \frac{d}{dx}(\sec^{-1} u) = \frac{du}{</td>
</tr>
<tr>
<td>6.</td>
<td>( \frac{d}{dx}(\csc^{-1} u) = -\frac{du}{</td>
</tr>
</tbody>
</table>

### Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4. The formulas are readily verified by differentiating the functions on the right-hand sides.

### TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant \( a \neq 0 \).

<table>
<thead>
<tr>
<th></th>
<th>Integral Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C ) (Valid for ( u^2 &lt; a^2 ))</td>
</tr>
<tr>
<td>2.</td>
<td>( \int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C ) (Valid for all ( u ))</td>
</tr>
<tr>
<td>3.</td>
<td>( \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left</td>
</tr>
</tbody>
</table>

The derivative formulas in Table 7.3 have \( a = 1 \), but in most integrations \( a \neq 1 \), and the formulas in Table 7.4 are more useful.

### EXAMPLE 11 Using the Integral Formulas

(a) \( \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \)
7.7 Inverse Trigonometric Functions

(b) \[ \int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} x \bigg|_0^1 = \tan^{-1} (1) - \tan^{-1} (0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \]

(c) \[ \int_{\sqrt{2}}^{\sqrt{2}/3} \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x \bigg|_{\sqrt{2}}^{\sqrt{2}/3} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \]

EXAMPLE 12 Using Substitution and Table 7.4

(a) \[ \int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{dx}{\sqrt{3^2 - x^2}} = \sin^{-1} \left( \frac{x}{3} \right) + C \]

(b) \[ \int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{a^2 - u^2}} \]

\[ = \frac{1}{2} \sin^{-1} \left( \frac{u}{a} \right) + C \]

\[ = \frac{1}{2} \sin^{-1} \left( \frac{2x}{\sqrt{3}} \right) + C \]

EXAMPLE 13 Completing the Square

Evaluate \[ \int \frac{dx}{\sqrt{4x - x^2}}. \]

Solution The expression \( \sqrt{4x - x^2} \) does not match any of the formulas in Table 7.4, so we first rewrite \( 4x - x^2 \) by completing the square:

\[ 4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2. \]

Then we substitute \( a = 2, u = x - 2, \) and \( du = dx \) to get

\[ \int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \]

\[ = \int \frac{du}{\sqrt{a^2 - u^2}} \]

\[ = \sin^{-1} \left( \frac{u}{a} \right) + C \]

\[ = \sin^{-1} \left( \frac{x - 2}{2} \right) + C \]

EXAMPLE 14 Completing the Square

Evaluate \[ \int \frac{dx}{4x^2 + 4x + 2}. \]
We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4(x^2 + x + \frac{1}{4}) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2} \int \frac{du}{u^2 + a^2}$$

$$a = 1, u = 2x + 1,$$

and $du/2 = dx$

$$= \frac{1}{2} \tan^{-1}\left(\frac{u}{a}\right) + C$$

Table 7.4, Formula 2

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C$$

$\blacksquare$

**EXAMPLE 15** Using Substitution

Evaluate

$$\int \frac{dx}{\sqrt{e^{2x} - 6}}.$$

**Solution**

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$$

$u = e^x$, $du = e^x \, dx$

$$dx = \frac{du}{u} = \frac{du}{e^x},$$

$a = \sqrt{6}$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C$$

Table 7.4, Formula 3

$$= \frac{1}{\sqrt{6}} \sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

$\blacksquare$