EXERCISES 8.4

Products of Powers of Sines and Cosines
Evaluate the integrals in Exercises 1–14.

1. \( \int_{0}^{\pi/2} \sin^3 x \, dx \)
2. \( \int_{0}^{\pi} \sin^3 \frac{x}{2} \, dx \)
3. \( \int_{-\pi/2}^{\pi/2} \cos^3 x \, dx \)
4. \( \int_{0}^{\pi/6} 3 \cos^5 3x \, dx \)
5. \( \int_{0}^{\pi/2} \sin^7 y \, dy \)
6. \( \int_{0}^{\pi/2} 7 \cos^3 t \, dt \)
Evaluate the integrals in Exercises 15–22.

15. \[ \int_{0}^{2\pi} \sqrt{1 - \cos x} \, dx \]
16. \[ \int_{0}^{\pi} \frac{1 - \cos 2x}{2} \, dx \]
17. \[ \int_{0}^{\pi} \sqrt{1 - \sin^2 t} \, dt \]
18. \[ \int_{0}^{\pi} 1 - \cos^2 \theta \, d\theta \]
19. \[ \int_{\pi/4}^{\pi/4} \sqrt{1 + \tan^2 x} \, dx \]
20. \[ \int_{\pi/4}^{\pi/4} \sqrt{\sec^2 x - 1} \, dx \]
21. \[ \int_{0}^{\pi} 0 \sqrt{1 - \cos 2\theta} \, d\theta \]
22. \[ \int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt \]

Evaluate the integrals in Exercises 23–32.

23. \[ \int_{-\pi/3}^{\pi/3} 2 \sec^3 x \, dx \]
24. \[ \int_{0}^{\pi/4} e^x \sec^3 e^x \, dx \]
25. \[ \int_{0}^{\pi/4} \sec^4 \theta \, d\theta \]
26. \[ \int_{0}^{\pi/12} 3 \sec^4 3x \, dx \]
27. \[ \int_{0}^{\pi/4} \csc^4 \theta \, d\theta \]
28. \[ \int_{0}^{\pi/2} 3 \csc^4 \frac{\theta}{2} \, d\theta \]
29. \[ \int_{0}^{\pi/4} 4 \tan^3 x \, dx \]
30. \[ \int_{0}^{\pi/4} 6 \tan^3 x \, dx \]
31. \[ \int_{\pi/6}^{\pi/3} \cot^3 x \, dx \]
32. \[ \int_{\pi/2}^{\pi/2} 8 \cot^4 t \, dt \]

Evaluate the integrals in Exercises 33–38.

33. \[ \int_{-\pi}^{\pi} \sin 3x \cos 2x \, dx \]
34. \[ \int_{0}^{\pi/2} \sin 2x \cos 3x \, dx \]

35. \[ \int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx \]
36. \[ \int_{0}^{\pi/2} \sin x \cos x \, dx \]
37. \[ \int_{0}^{\pi} \cos 3x \cos 4x \, dx \]
38. \[ \int_{\pi/2}^{\pi/2} \cos x \cos 7x \, dx \]

Theory and Examples

39. Surface area Find the area of the surface generated by revolving the arc

\[ x = t^{2/3}, \quad y = t^{1/2}, \quad 0 \leq t \leq 2, \]

about the x-axis.

40. Arc length Find the length of the curve

\[ y = \ln(\cos x), \quad 0 \leq x \leq \pi/3. \]

41. Arc length Find the length of the curve

\[ y = \ln(\sec x), \quad 0 \leq x \leq \pi/4. \]

42. Center of gravity Find the center of gravity of the region bounded by the x-axis, the curve \( y = \sec x \), and the lines \( x = -\pi/4, x = \pi/4 \).

43. Volume Find the volume generated by revolving one arch of the curve \( y = \sin x \) about the x-axis.

44. Area Find the area between the x-axis and the curve \( y = \sqrt{1 + \cos 4x} \), \( 0 \leq x \leq \pi \).

45. Orthogonal functions Two functions \( f \) and \( g \) are said to be orthogonal on an interval \( a \leq x \leq b \) if

\[ \int_{a}^{b} f(x)g(x) \, dx = 0. \]

a. Prove that \( \sin mx \) and \( \sin nx \) are orthogonal on any interval of length \( 2\pi \) provided \( m \) and \( n \) are integers such that \( m^2 \neq n^2 \).

b. Prove the same for \( \cos mx \) and \( \cos nx \).

c. Prove the same for \( \sin mx \) and \( \cos nx \) even if \( m = n \).

46. Fourier series A finite Fourier series is given by the sum

\[ f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx \]

Show that the \( m \)th coefficient \( a_m \) is given by the formula

\[ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx. \]