Improper Integrals

Up to now, definite integrals have been required to have two properties. First, that the domain of integration \([a, b]\) be finite. Second, that the range of the integrand be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve from to is an example for which the domain is infinite (Figure 8.17a). The integral for the area under the curve of between and is an example for which the range of the integrand is infinite (Figure 8.17b). In either case, the integrals are said to be improper and are calculated as limits. We will see that improper integrals play an important role when investigating the convergence of certain infinite series in Chapter 11.

**Infinite Limits of Integration**

Consider the infinite region that lies under the curve \(y = e^{-x^2}\) in the first quadrant (Figure 8.18a). You might think this region has infinite area, but we will see that the natural value to assign is finite. Here is how to assign a value to the area. First find the area \(A(b)\) of the portion of the region that is bounded on the right by \(x = b\) (Figure 8.18b).

\[
A(b) = \int_{0}^{b} e^{-x^2} \, dx = -2e^{-b^2/2} \bigg|_{0}^{b} = -2e^{-b^2/2} + 2
\]

Then find the limit of \(A(b)\) as \(b \to \infty\)

\[
\lim_{b \to \infty} A(b) = \lim_{b \to \infty} (-2e^{-b^2/2} + 2) = 2.
\]
The value we assign to the area under the curve from 0 to $\infty$ is
\[ \int_0^\infty e^{-x^2} \, dx = \lim_{b \to \infty} \int_0^b e^{-x^2} \, dx = 2. \]

**DEFINITION** Type I Improper Integrals
Integrals with infinite limits of integration are improper integrals of Type I.

1. If $f(x)$ is continuous on $[a, \infty)$, then
   \[ \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx. \]

2. If $f(x)$ is continuous on $(-\infty, b]$, then
   \[ \int_{-\infty}^b f(x) \, dx = \lim_{a \to -\infty} \int_a^b f(x) \, dx. \]

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
   \[ \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^\infty f(x) \, dx, \]
   where $c$ is any real number.

   In each case, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

It can be shown that the choice of $c$ in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^\infty f(x) \, dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.18 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is infinite.

**EXAMPLE 1** Evaluating an Improper Integral on $[1, \infty)$
Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

**Solution** We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to $b$ is
\[ \int_1^b \frac{\ln x}{x^2} \, dx = \left[ \frac{\ln x}{x} \right]_1^b - \int_1^b \frac{1}{x^2} \left( \frac{1}{x} \right) \, dx \]
\[ = -\ln b - \frac{1}{b} - \left[ \frac{1}{x} \right]_1^b \]
\[ = -\ln b - \frac{1}{b} + 1. \]
The limit of the area as \( b \to \infty \) is

\[
\int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x^2} \, dx
\]

\[
= \lim_{b \to \infty} \left[ -\frac{\ln b}{b} + \frac{1}{b} + 1 \right]
\]

\[
= - \left( \lim_{b \to \infty} \frac{\ln b}{b} \right) - 0 + 1
\]

\[
= - \left[ \lim_{b \to \infty} \frac{1/b}{1} \right] + 1 = 0 + 1 = 1.
\]

Thus, the improper integral converges and the area has finite value 1.

**EXAMPLE 2** Evaluating an Integral on \((-\infty, \infty)\)

Evaluate

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.
\]

**Solution** According to the definition (Part 3), we can write

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{0}^{\infty} \frac{dx}{1 + x^2} + \int_{0}^{-\infty} \frac{dx}{1 + x^2}.
\]

Next we evaluate each improper integral on the right side of the equation above.

\[
\int_{0}^{\infty} \frac{dx}{1 + x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2}
\]

\[
= \lim_{a \to -\infty} \tan^{-1} x \bigg|_{a}^{0}
\]

\[
= \lim_{a \to -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left( -\frac{\pi}{2} \right) = \frac{\pi}{2}
\]

\[
\int_{0}^{\infty} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1 + x^2}
\]

\[
= \lim_{b \to \infty} \tan^{-1} x \bigg|_{0}^{b}
\]

\[
= \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\]

Thus,

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\]

Since \( 1/(1 + x^2) > 0 \), the improper integral can be interpreted as the (finite) area beneath the curve and above the \( x \)-axis (Figure 8.20).
The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

**EXAMPLE 3  Determining Convergence**

For what values of $p$ does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

**Solution** If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1} \bigg|_1^b = \frac{1}{1-p} (b^{-p+1} - 1) = \frac{1}{1-p} (\frac{1}{b^{p-1}} - 1).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p} = \lim_{b \to \infty} \left[ \frac{1}{1-p} (\frac{1}{b^{p-1}} - 1) \right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b \to \infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b \to \infty} \int_1^b \frac{dx}{x} = \lim_{b \to \infty} \ln x \bigg|_1^b = \lim_{b \to \infty} (\ln b - \ln 1) = \infty.$$ 

**Integrands with Vertical Asymptotes**

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand $f$ is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of $f$ and above the $x$-axis between the limits of integration.
Consider the region in the first quadrant that lies under the curve \( y = \frac{1}{\sqrt{x}} \) from \( x = 0 \) to \( x = 1 \) (Figure 8.17b). First we find the area of the portion from \( a \) to 1 (Figure 8.21).

\[
\int_{a}^{1} \frac{dx}{\sqrt{x}} = 2 \sqrt{x} \bigg|_{a}^{1} = 2 - 2 \sqrt{a}
\]

Then we find the limit of this area as \( a \to 0^{+} \):

\[
\lim_{a \to 0^{+}} \int_{a}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^{+}} (2 - 2 \sqrt{a}) = 2.
\]

The area under the curve from 0 to 1 is finite and equals

\[
\int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^{+}} \int_{a}^{1} \frac{dx}{\sqrt{x}} = 2.
\]

**DEFINITION** Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1. If \( f(x) \) is continuous on \((a, b]\) and is discontinuous at \( a \) then

\[
\int_{a}^{b} f(x) \, dx = \lim_{c \to a^{+}} \int_{c}^{b} f(x) \, dx.
\]

2. If \( f(x) \) is continuous on \([a, b)\) and is discontinuous at \( b \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{c \to b^{-}} \int_{a}^{c} f(x) \, dx.
\]

3. If \( f(x) \) is discontinuous at \( c \), where \( a < c < b \), and continuous on \([a, c) \cup (c, b]\), then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
\]

In each case, if the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges.

In Part 3 of the definition, the integral on the left side of the equation converges if both integrals on the right side converge; otherwise it diverges.

**EXAMPLE 4** A Divergent Improper Integral

Investigate the convergence of

\[
\int_{0}^{1} \frac{1}{1 - x} \, dx.
\]
Solution The integrand \( f(x) = 1/(1-x) \) is continuous on \([0, 1)\) but is discontinuous at \( x = 1 \) and becomes infinite as \( x \to 1^- \) (Figure 8.22). We evaluate the integral as
\[
\lim_{b \to 1^-} \int_0^b \frac{1}{1-x} \, dx = \lim_{b \to 1^-} \left[ -\ln |1-x| \right]_0^b
\]
\[
= \lim_{b \to 1^-} [-\ln (1-b) + 0] = \infty.
\]
The limit is infinite, so the integral diverges.

**EXAMPLE 5** Vertical Asymptote at an Interior Point
Evaluate
\[
\int_0^3 \frac{dx}{x-1}.
\]

**Solution** The integrand has a vertical asymptote at \( x = 1 \) and is continuous on \([0, 1)\) and \((1, 3]\) (Figure 8.23). Thus, by Part 3 of the definition above,
\[
\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.
\]
Next, we evaluate each improper integral on the right-hand side of this equation.
\[
\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}
\]
\[
= \lim_{b \to 1^-} 3(x-1)^{1/3} \bigg|_0^b
\]
\[
= \lim_{b \to 1^-} [3(b-1)^{1/3} + 3] = 3
\]
\[
\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}
\]
\[
= \lim_{c \to 1^+} 3(x-1)^{1/3} \bigg|_c^3
\]
\[
= \lim_{c \to 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt{2}
\]
We conclude that
\[
\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt{2}.
\]

**EXAMPLE 6** A Convergent Improper Integral
Evaluate
\[
\int_2^\infty \frac{x + 3}{(x-1)(x^2 + 1)} \, dx.
\]
Solution

\[ \int_{2}^{\infty} \frac{x + 3}{(x - 1)(x^2 + 1)} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{x + 3}{(x - 1)(x^2 + 1)} \, dx \]

\[ = \lim_{b \to \infty} \int_{2}^{b} \left( \frac{2}{x - 1} - \frac{2x + 1}{x^2 + 1} \right) \, dx \quad \text{Partial fractions} \]

\[ = \lim_{b \to \infty} \left[ 2 \ln(x - 1) - \ln(x^2 + 1) - \tan^{-1} x \right]_{2}^{b} \]

\[ = \lim_{b \to \infty} \left[ \ln \left( \frac{x - 1}{x^2 + 1} \right) - \tan^{-1} x \right]_{2}^{b} \quad \text{Combine the logarithms.} \]

\[ = \lim_{b \to \infty} \left[ \ln \left( \frac{(b - 1)^2}{b^2 + 1} \right) - \tan^{-1} b \right] - \ln \left( \frac{1}{5} \right) + \tan^{-1} 2 \]

\[ = 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1} 2 \approx 1.1458 \]

Notice that we combined the logarithms in the antiderivative before we calculated the limit as \( b \to \infty \). Had we not done so, we would have encountered the indeterminate form

\[ \lim_{b \to \infty} (2 \ln(b - 1) - \ln(b^2 + 1)) = \infty - \infty. \]

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. \( \square \)

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral in Example 6 using Maple, enter

\[ > f := (x + 3)/(x - 1)*(x^2 + 1); \]

Then use the integration command

\[ > \text{int}(f, x = 2..\infty); \]

Maple returns the answer

\[ -\frac{1}{2} \pi + \ln 5 + \arctan 2. \]

To obtain a numerical result, use the evaluation command \texttt{evalf} and specify the number of digits, as follows:

\[ > \text{evalf}(%); \]

The symbol \% instructs the computer to evaluate the last expression on the screen, in this case \((-1/2) \pi + \ln 5 + \arctan 2\). Maple returns 1.14579.

Using Mathematica, entering

\[ \text{In}[1]:= \text{Integrate} \left[ (x + 3)/(x - 1)(x^2 + 1), \{x, 2, \infty\} \right] \]

returns

\[ \text{Out}[1]= -\frac{\text{Pi}}{2} + \text{ArcTan}[2] + \text{Log}[5]. \]

To obtain a numerical result with six digits, use the command “\text{N[\%}, 6\text{]}; it also yields 1.14579.
EXAMPLE 7  Finding the Volume of an Infinite Solid

The cross-sections of the solid horn in Figure 8.24 perpendicular to the $x$-axis are circular disks with diameters reaching from the $x$-axis to the curve $y = e^x$, $-\infty < x \leq \ln 2$. Find the volume of the horn.

Solution  The area of a typical cross-section is

\[ A(x) = \pi (\text{radius})^2 = \pi \left( \frac{1}{2} y \right)^2 = \frac{\pi}{4} e^{2x}. \]

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from $b$ to $\ln 2$. As in Section 6.1 (the method of slicing), the volume of this portion is

\[ V = \int_b^{\ln 2} A(x) \, dx = \int_b^{\ln 2} \frac{\pi}{4} e^{2x} \, dx = \frac{\pi}{8} e^{2x} \bigg|_b^{\ln 2} = \frac{\pi}{8} (e^{2\ln 2} - e^{2b}) = \frac{\pi}{8} (4 - e^{2b}). \]

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4 - 0) = \pi/2$. The volume of the horn is $\pi/2$. ■

EXAMPLE 8  An Incorrect Calculation

Evaluate

\[ \int_0^3 \frac{dx}{x - 1}. \]

Solution  Suppose we fail to notice the discontinuity of the integrand at $x = 1$, interior to the interval of integration. If we evaluate the integral as an ordinary integral we get

\[ \int_0^3 \frac{dx}{x - 1} = \left. \ln |x - 1| \right|_0^3 = \ln 2 - \ln 1 = \ln 2. \]

This result is wrong because the integral is improper. The correct evaluation uses limits:

\[ \int_0^3 \frac{dx}{x - 1} = \int_0^1 \frac{dx}{x - 1} + \int_1^3 \frac{dx}{x - 1}, \]

where

\[ \int_0^1 \frac{dx}{x - 1} = \lim_{b \to 1^-} \int_0^b \frac{dx}{x - 1} = \lim_{b \to 1^-} \ln |x - 1|_0^b = \lim_{b \to 1^-} (\ln |b - 1| - \ln |1|) = \lim_{b \to 1^-} \ln (1 - b) = -\infty. \]

Since $\int_0^1 dx/(x - 1)$ is divergent, the original integral $\int_0^3 dx/(x - 1)$ is divergent. ■

Example 8 illustrates what can go wrong if you mistake an improper integral for an ordinary integral. Whenever you encounter an integral $\int_a^b f(x) \, dx$ you must examine the function $f$ on $[a, b]$ and then decide if the integral is improper. If $f$ is continuous on $[a, b]$, it will be proper, an ordinary integral.
Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that’s the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

EXAMPLE 9 Investigating Convergence

Does the integral converge?

Solution

By definition,

We cannot evaluate the latter integral directly because it is nonelementary. But we can show that its limit as is finite. We know that is an increasing function of . Therefore either it becomes infinite as or it has a finite limit as . It does not become infinite: For every value of we have (Figure 8.25), so that

Hence

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 4.

The comparison of and in Example 9 is a special case of the following test.

Theorem 1 Direct Comparison Test

Let and be continuous on such that for all . Then

1. converges if converges

2. diverges if diverges.

The reasoning behind the argument establishing Theorem 1 is similar to that in Example 9.

If for then

The historical biography of Karl Weierstrass (1815–1897) is also included in the text.
From this it can be argued, as in Example 9, that

\[ \int_a^\infty f(x) \, dx \text{ converges if } \int_a^\infty g(x) \, dx \text{ converges.} \]

Turning this around says that

\[ \int_a^\infty g(x) \, dx \text{ diverges if } \int_a^\infty f(x) \, dx \text{ diverges.} \]

**EXAMPLE 10** Using the Direct Comparison Test

(a) \[ \int_1^\infty \frac{\sin^2 x}{x^2} \, dx \text{ converges because} \]

\[ 0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x^2} \, dx \text{ converges.} \]  

Example 3

(b) \[ \int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} \, dx \text{ diverges because} \]

\[ \frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x} \, dx \text{ diverges.} \]  

Example 3

---

**THEOREM 2** Limit Comparison Test

If the positive functions \( f \) and \( g \) are continuous on \([a, \infty)\) and if

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty, \]

then

\[ \int_a^\infty f(x) \, dx \quad \text{and} \quad \int_a^\infty g(x) \, dx \]

both converge or both diverge.

---

A proof of Theorem 2 is given in advanced calculus. Although the improper integrals of two functions from \( a \) to \( \infty \) may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.
EXAMPLE 11 Using the Limit Comparison Test

Show that

\[ \int_1^\infty \frac{dx}{1 + x^2} \]

converges by comparison with \( \int_1^\infty (1/x^2) \, dx \). Find and compare the two integral values.

Solution The functions \( f(x) = 1/x^2 \) and \( g(x) = 1/(1 + x^2) \) are positive and continuous on \([1, \infty)\). Also,

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1/x^2}{1/(1 + x^2)} = \lim_{x \to \infty} \frac{1 + x^2}{x^2} = \lim_{x \to \infty} \left( \frac{1}{x^2} + 1 \right) = 0 + 1 = 1,
\]

a positive finite limit (Figure 8.26). Therefore, \( \int_1^\infty \frac{dx}{1 + x^2} \) converges because \( \int_1^\infty \frac{dx}{x^2} \) converges.

The integrals converge to different values, however.

\[
\int_1^\infty \frac{dx}{x^2} = \frac{1}{2} - 1 = 1 \quad \text{Example 3}
\]

and

\[
\int_1^\infty \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{1 + x^2} = \lim_{b \to \infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\]

EXAMPLE 12 Using the Limit Comparison Test

Show that

\[ \int_1^\infty \frac{3}{e^x + 5} \, dx \]

converges.

Solution From Example 9, it is easy to see that \( \int_1^\infty e^{-x} \, dx = \int_1^\infty (1/e^x) \, dx \) converges. Moreover, we have

\[
\lim_{x \to \infty} \frac{1/e^x}{3/(e^x + 5)} = \lim_{x \to \infty} \frac{e^x + 5}{3e^x} = \lim_{x \to \infty} \left( \frac{1}{3} + \frac{5}{3e^x} \right) = \frac{1}{3},
\]

a positive finite limit. As far as the convergence of the improper integral is concerned, \( 3/(e^x + 5) \) behaves like \( 1/e^x \).

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley
Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION: TYPE I

1. Upper limit

$$\int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x^2} \, dx$$

2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1 + x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2}$$

3. Both limits

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{1 + x^2} + \lim_{c \to \infty} \int_{0}^{c} \frac{dx}{1 + x^2}$$

INTEGRAND BECOMES INFINITE: TYPE II

4. Upper endpoint

$$\int_{0}^{1} \frac{dx}{(x - 1)^{2/3}} = \lim_{b \to 1} \int_{0}^{b} \frac{dx}{(x - 1)^{2/3}}$$

5. Lower endpoint

$$\int_{1}^{3} \frac{dx}{(x - 1)^{2/3}} = \lim_{b \to 1} \int_{1}^{b} \frac{dx}{(x - 1)^{2/3}}$$

6. Interior point

$$\int_{0}^{3} \frac{dx}{(x - 1)^{2/3}} = \int_{0}^{1} \frac{dx}{(x - 1)^{2/3}} + \int_{1}^{3} \frac{dx}{(x - 1)^{2/3}}$$