This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

**Area in the Plane**

The region \(OTS\) in Figure 10.48 is bounded by the rays \(\theta = \alpha\) and \(\theta = \beta\) and the curve \(r = f(\theta)\). We approximate the region with \(n\) nonoverlapping fan-shaped circular sectors based on a partition \(P\) of angle \(TOS\). The typical sector has radius \(r_k = f(\theta_k)\) and central angle of radian measure \(\Delta \theta_k\). Its area is \(\Delta \theta_k/2\pi\) times the area of a circle of radius \(r_k\), or

\[
A_k = \frac{1}{2} r_k^2 \Delta \theta_k = \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta \theta_k.
\]

The area of region \(OTS\) is approximately

\[
\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta \theta_k.
\]
If \( f \) is continuous, we expect the approximations to improve as the norm of the partition \( ||P|| \to 0 \), and we are led to the following formula for the region’s area:

\[
A = \lim_{||P|| \to 0} \sum_{k=1}^{n} \frac{1}{2} (f(\theta_k))^2 \Delta \theta_k
\]

\[
= \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta.
\]

Area of the Fan-Shaped Region Between the Origin and the Curve
\( r = f(\theta) \), \( \alpha \leq \theta \leq \beta \)

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.
\]

This is the integral of the area differential (Figure 10.49)

\[
dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta.
\]

**EXAMPLE 1** Finding Area
Find the area of the region in the plane enclosed by the cardioid \( r = 2(1 + \cos \theta) \).

**Solution** We graph the cardioid (Figure 10.50) and determine that the radius \( OP \) sweeps out the region exactly once as \( \theta \) runs from 0 to \( 2\pi \). The area is therefore

\[
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2} \cdot 4(1 + \cos \theta)^2 \, d\theta
\]

\[
= \int_{0}^{2\pi} 2(1 + 2 \cos \theta + \cos^2 \theta) \, d\theta
\]

\[
= \int_{0}^{2\pi} \left( 2 + 4 \cos \theta + 2 \frac{1 + \cos 2\theta}{2} \right) \, d\theta
\]

\[
= \int_{0}^{2\pi} \left( 3 + 4 \cos \theta + \cos 2\theta \right) \, d\theta
\]

\[
= \left[ 3\theta + 4 \sin \theta + \frac{\sin 2\theta}{2} \right]_{0}^{2\pi} = 6\pi - 0 = 6\pi.
\]

**EXAMPLE 2** Finding Area
Find the area inside the smaller loop of the limaçon

\( r = 2 \cos \theta + 1 \).

**Solution** After sketching the curve (Figure 10.51), we see that the smaller loop is traced out by the point \( (r, \theta) \) as \( \theta \) increases from \( \theta = 2\pi/3 \) to \( \theta = 4\pi/3 \). Since the curve is symmetric about the \( x \)-axis (the equation is unaltered when we replace \( \theta \) by \( -\theta \)), we may calculate the area of the shaded half of the inner loop by integrating from \( \theta = 2\pi/3 \) to \( \theta = \pi \). The area we seek will be twice the resulting integral:

\[
A = 2 \int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta.
\]
Since
\[ r^2 = (2 \cos \theta + 1)^2 = 4 \cos^2 \theta + 4 \cos \theta + 1 \]
\[ = 4 \cdot \frac{1 + \cos 2\theta}{2} + 4 \cos \theta + 1 \]
\[ = 2 + 2 \cos 2\theta + 4 \cos \theta + 1 \]
\[ = 3 + 2 \cos 2\theta + 4 \cos \theta, \]
we have
\[ A = \int_{\pi/3}^{\pi} (3 + 2 \cos 2\theta + 4 \cos \theta) \, d\theta \]
\[ = \left[ 3\theta + \sin 2\theta + 4 \sin \theta \right]_{\pi/3}^{\pi} \]
\[ = (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4 \cdot \frac{\sqrt{3}}{2} \right) \]
\[ = \pi - \frac{3\sqrt{3}}{2}. \]

To find the area of a region like the one in Figure 10.52, which lies between two polar curves \( r_1 = r_1(\theta) \) and \( r_2 = r_2(\theta) \) from \( \theta = \alpha \) to \( \theta = \beta \), we subtract the integral of \((1/2)r_2^2 \, d\theta\) from the integral of \((1/2)r_1^2 \, d\theta\). This leads to the following formula.

\[
\text{Area of the Region } 0 \leq r_1(\theta) \leq r \leq r_2(\theta), \quad \alpha \leq \theta \leq \beta
\]
\[ A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \quad (1) \]

**EXAMPLE 3** Finding Area Between Polar Curves

Find the area of the region that lies inside the circle \( r = 1 \) and outside the cardioid \( r = 1 - \cos \theta \).

**Solution** We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is \( r_2 = 1 \), the inner curve is \( r_1 = 1 - \cos \theta \), and \( \theta \) runs from \(-\pi/2\) to \(\pi/2\). The area, from Equation (1), is
\[ A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \]
\[ = 2 \int_{0}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \quad \text{Symmetry} \]
\[ = \int_{0}^{\pi/2} (1 - (1 - 2 \cos \theta + \cos^2 \theta)) \, d\theta \]
\[ = \int_{0}^{\pi/2} (2 \cos \theta - \cos^2 \theta) \, d\theta = \int_{0}^{\pi/2} \left(2 \cos \theta - \frac{1 + \cos 2\theta}{2} \right) \, d\theta \]
\[ = \left[ 2 \sin \theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}. \]
Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve \( r = f(\theta), \alpha \leq \theta \leq \beta \), by parametrizing the curve as

\[
x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta, \quad \alpha \leq \theta \leq \beta.
\]  

(2)

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta.
\]

This equation becomes

\[
L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]

when Equations (2) are substituted for \( x \) and \( y \) (Exercise 33).

### EXAMPLE 4  Finding the Length of a Cardioid

Find the length of the cardioid \( r = 1 - \cos \theta \).

#### Solution

We sketch the cardioid to determine the limits of integration (Figure 10.54). The point \( P(r, \theta) \) traces the curve once, counterclockwise as \( \theta \) runs from 0 to \( 2\pi \), so these are the values we take for \( \alpha \) and \( \beta \).

With

\[
r = 1 - \cos \theta, \quad \frac{dr}{d\theta} = \sin \theta,
\]

we have

\[
r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos \theta)^2 + (\sin \theta)^2
\]

\[
= 1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta = 2 - 2 \cos \theta
\]

and

\[
L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta
\]

\[
= \int_{0}^{2\pi} \sqrt{4 \sin^2 \frac{\theta}{2}} \, d\theta = \int_{0}^{2\pi} 2 \sin \frac{\theta}{2} \, d\theta
\]

\[
= \left[-4 \cos \frac{\theta}{2}\right]_{0}^{2\pi} = -4 \cos \pi + 4 \cos 0 = 8
\]

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To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve with Equations (2) and apply the surface area equations for parametrized curves in Section 6.5.

\[
r = f(s), \quad a \leq s \leq b, \quad r = c - 4 \cos^2 \theta \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi
\]

\[
\left[ -4 \cos \frac{\theta}{2} \right]_{\theta=0}^{2\pi} = 4 + 4 = 8.
\]

**Area of a Surface of Revolution**

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve \( r = f(\theta), \alpha \leq \theta \leq \beta \), with Equations (2) and apply the surface area equations for parametrized curves in Section 6.5.

**EXAMPLE 5 Finding Surface Area**

Find the area of the surface generated by revolving the right-hand loop of the lemniscate \( r^2 = \cos 2\theta \) about the \( y \)-axis.

**Solution** We sketch the loop to determine the limits of integration (Figure 10.55a). The point \( P(r, \theta) \) traces the curve once, counterclockwise as \( \theta \) runs from \(-\pi/4\) to \(\pi/4\), so these are the values we take for \( \alpha \) and \( \beta \).

We evaluate the area integrand in Equation (5) in stages. First,

\[
2\pi r \cos \theta \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} = 2\pi \cos \theta \sqrt{r^4 + \left( \frac{dr}{d\theta} \right)^2}.
\]

Next, \( r^2 = \cos 2\theta \), so

\[
2r \frac{dr}{d\theta} = -2 \sin 2\theta
\]

\[
r \frac{dr}{d\theta} = -\sin 2\theta
\]

\[
\left( r \frac{dr}{d\theta} \right)^2 = \sin^2 2\theta.
\]
Finally, \( r^4 = (r^2)^2 = \cos^2 2\theta \), so the square root on the right-hand side of Equation (6) simplifies to

\[
\sqrt{r^4 + \left( r \frac{dr}{d\theta} \right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.
\]

All together, we have

\[
S = \int_{\alpha}^{\beta} 2\pi r \cos \theta \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \quad \text{Equation (5)}
\]

\[
= \int_{-\pi/4}^{\pi/4} 2\pi \cos \theta \cdot (1) \, d\theta
\]

\[
= 2\pi \left[ \sin \theta \right]_{-\pi/4}^{\pi/4}
\]

\[
= 2\pi \left[ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right] = 2\pi \sqrt{2}.
\]