Determining Convergence or Divergence

Which of the series in Exercises 1–36 converge, and which diverge?
Give reasons for your answers.

1. \[ \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt{n}} \]
2. \[ \sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}} \]
3. \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \]
4. \[ \sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2} \]
5. \[ \sum_{n=1}^{\infty} \frac{2n}{3n - 1} \]
6. \[ \sum_{n=1}^{\infty} \frac{n + 1}{n^2 \sqrt{n}} \]
7. \[ \sum_{n=1}^{\infty} \left( \frac{n}{3n + 1} \right)^n \]
8. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2}} \]
9. \[ \sum_{n=1}^{\infty} \frac{1}{n \ln (\ln n)} \]
10. \[ \sum_{n=1}^{\infty} \left( \frac{n}{\ln n} \right)^2 \]
11. \[ \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3} \]
12. \[ \sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^3} \]
13. \[ \sum_{n=1}^{\infty} \frac{1}{(1 + \ln n)^2} \]
14. \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \]
15. \[ \sum_{n=1}^{\infty} \frac{1}{n + 1} \]
16. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \ln n \]
17. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 1} \]
18. \[ \sum_{n=1}^{\infty} \frac{1 - n}{n^2} \]
19. \[ \sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n} \]
20. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]
21. \[ \sum_{n=1}^{\infty} \frac{1 - 2^n}{n^2 n^2} \]
22. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \]
23. \[ \sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 1} \]
24. \[ \sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^n} \]
25. \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{n} \]
26. \[ \sum_{n=1}^{\infty} \frac{1}{n} \]
27. \[ \sum_{n=1}^{\infty} \frac{10n + 1}{n(n + 1)(n + 2)} \]
28. \[ \sum_{n=1}^{\infty} \frac{2n^3 - 3n}{n^2 (n - 2)(n^2 + 5)} \]
29. \[ \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1/3}} \]
30. \[ \sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1/3}} \]
31. \[ \sum_{n=1}^{\infty} \frac{\coth n}{n} \]
32. \[ \sum_{n=1}^{\infty} \frac{\tanh n}{n^2} \]
33. \[ \sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}} \]
34. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} \]
35. \[ \sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n} \]
36. \[ \sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2} \]

Theory and Examples

37. Prove (a) Part 2 and (b) Part 3 of the Limit Comparison Test.

38. If \( \sum_{n=1}^{\infty} a_n \) is a convergent series of nonnegative numbers, can anything be said about \( \sum_{n=1}^{\infty} (a_n/n) \)? Explain.

39. Suppose that \( a_n > 0 \) and \( b_n > 0 \) for \( n \geq N \) (\( N \) an integer). If \( \lim_{n \to \infty} (a_n/b_n) = \infty \) and \( \sum a_n \) converges, can anything be said about \( \sum b_n \)? Give reasons for your answer.

40. Prove that if \( \sum a_n \) is a convergent series of nonnegative terms, then \( \sum a_n^2 \) converges.

**COMPUTER EXPLORATION**

41. It is not yet known whether the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n} \]

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

a. Define the sequence of partial sums

\[ s_k = \sum_{n=1}^{k} \frac{1}{n^3 \sin^2 n} \]

What happens when you try to find the limit of \( s_k \) as \( k \to \infty \)? Does your CAS find a closed form answer for this limit?

b. Plot the first 100 points \((k, s_k)\) for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?

c. Next plot the first 200 points \((k, s_k)\). Discuss the behavior in your own words.

d. Plot the first 400 points \((k, s_k)\). What happens when \( k = 355 \)? Calculate the number 355/113. Explain from your calculation what happened at \( k = 355 \). For what values of \( k \) would you guess this behavior might occur again?