The Ratio and Root Tests

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio $a_{n+1}/a_n$. For a geometric series $\sum ar^n$, this rate is a constant $((ar^{n+1})/(ar^n) = r)$, and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result. We prove it on the next page using the Comparison Test.
Proof

(a) \( \rho < 1 \). Let \( r \) be a number between \( \rho \) and 1. Then the number \( \epsilon = r - \rho \) is positive. Since

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho,
\]

\( a_{n+1}/a_n \) must lie within \( \epsilon \) of \( \rho \) when \( n \) is large enough, say for all \( n \geq N \). In particular

\[
\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{when} \quad n \geq N.
\]

That is,

\[
\begin{align*}
    a_{N+1} &< ra_N, \\
    a_{N+2} &< ra_{N+1} < r^2a_N, \\
    a_{N+3} &< ra_{N+2} < r^3a_N,
    \vdots \\
    a_{N+m} &< ra_{N+m-1} < r^mA_N.
\end{align*}
\]

These inequalities show that the terms of our series, after the \( N \)th term, approach zero more rapidly than the terms in a geometric series with ratio \( r < 1 \). More precisely, consider the series \( \sum c_n \), where \( c_n = a_n \) for \( n = 1, 2, \ldots, N \) and \( c_{N+1} = ra_N, c_{N+2} = r^2a_N, \ldots, c_{N+m} = r^mA_N, \ldots \). Now \( a_n \leq c_n \) for all \( n \), and

\[
\sum_{n=1}^{\infty} c_n = a_1 + a_2 + \cdots + a_{N-1} + a_N + ra_N + r^2a_N + \cdots = a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots).
\]

The geometric series \( 1 + r + r^2 + \cdots \) converges because \( |r| < 1 \), so \( \sum c_n \) converges. Since \( a_n \leq c_n \), \( \sum a_n \) also converges.

(b) \( 1 < \rho \leq \infty \). From some index \( M \) on,

\[
\begin{align*}
    a_{n+1}/a_n &> 1 \quad \text{and} \quad a_M < a_{M+1} < a_{M+2} < \cdots.
\end{align*}
\]

The terms of the series do not approach zero as \( n \) becomes infinite, and the series diverges by the \( n \)th-Term Test.
(c) \( \rho = 1 \). The two series
\[
\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
show that some other test for convergence must be used when \( \rho = 1 \).

For \( \sum_{n=1}^{\infty} \frac{1}{n} \):
\[
\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \rightarrow 1.
\]

For \( \sum_{n=1}^{\infty} \frac{1}{n^2} \):
\[
\frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left( \frac{n}{n+1} \right)^2 \rightarrow 1^2 = 1.
\]

In both cases, \( \rho = 1 \), yet the first series diverges, whereas the second converges.

**EXAMPLE 1** Applying the Ratio Test

Investigate the convergence of the following series.

(a) \( \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \)

(b) \( \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \)

(c) \( \sum_{n=1}^{\infty} \frac{4^n n!}{(2n)!} \)

**Solution**

(a) For the series \( \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \),
\[
a_{n+1} = \frac{2^{n+1} + 5}{3^{n+1}} = \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \frac{2 + \frac{5}{2^n}}{1 + \frac{5}{2^n}} \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.
\]

The series converges because \( \rho = 2/3 \) is less than 1. This does not mean that 2/3 is the sum of the series. In fact,
\[
\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - 2/3} + \frac{5}{1 - 1/3} = \frac{21}{2}.
\]

(b) If \( a_n = \frac{(2n)!}{n!n!} \), then \( a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!} \) and
\[
\frac{a_{n+1}}{a_n} = \frac{n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \cdot \frac{(n+1)!}{n!n!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4 \cdot \frac{n+2}{n+1} \rightarrow 4.
\]

The series diverges because \( \rho = 4 \) is greater than 1.

(c) If \( a_n = \frac{4^n n!}{(2n)!} \), then
\[
\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = 2 \cdot \frac{n+1}{2n+1} \rightarrow 1.
\]
Because the limit is \( \rho = 1 \), we cannot decide from the Ratio Test whether the series converges. When we notice that \( \frac{a_{n+1}}{a_n} = \frac{(2n + 2)}{(2n + 1)} \), we conclude that \( a_{n+1} \) is always greater than \( a_n \) because \( (2n + 2)/(2n + 1) \) is always greater than 1. Therefore, all terms are greater than or equal to \( a_1 = 2 \), and the \( n \)th term does not approach zero as \( n \to \infty \). The series diverges.

**The Root Test**

The convergence tests we have so far for \( \sum a_n \) work best when the formula for \( a_n \) is relatively simple. But consider the following.

**EXAMPLE 2** Let \( a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even} \end{cases} \). Does \( \sum a_n \) converge?

**Solution** We write out several terms of the series:

\[
\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \ldots
\]

Clearly, this is not a geometric series. The \( n \)th term approaches zero as \( n \to \infty \), so we do not know if the series diverges. The Integral Test does not look promising. The Ratio Test produces

\[
\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2n}, & n \text{ odd} \\ \frac{n + 1}{2}, & n \text{ even} \end{cases}
\]

As \( n \to \infty \), the ratio is alternately small and large and has no limit. A test that will answer the question (the series converges) is the Root Test.

**THEOREM 13** **The Root Test**

Let \( \sum a_n \) be a series with \( a_n \geq 0 \) for \( n \geq N \), and suppose that

\[
\lim_{n \to \infty} \sqrt[n]{a_n} = \rho.
\]

Then

(a) the series converges if \( \rho < 1 \),

(b) the series diverges if \( \rho > 1 \) or \( \rho \) is infinite,

(c) the test is inconclusive if \( \rho = 1 \).

**Proof**

(a) \( \rho < 1 \). Choose an \( \epsilon > 0 \) so small that \( \rho + \epsilon < 1 \). Since \( \sqrt[n]{a_n} \to \rho \), the terms \( \sqrt[n]{a_n} \) eventually get closer than \( \epsilon \) to \( \rho \). In other words, there exists an index \( M \geq N \) such that

\[
\sqrt[n]{a_n} < \rho + \epsilon \quad \text{when} \quad n \geq M.
\]
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Then it is also true that

\[ a_n < (\rho + e)^n \quad \text{for } n \geq M. \]

Now, \( \sum_{n=M}^{\infty} (\rho + e)^n \), a geometric series with ratio \((\rho + e) < 1\), converges. By comparison, \( \sum_{n=M}^{\infty} a_n \) converges, from which it follows that

\[ \sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{M-1} + \sum_{n=M}^{\infty} a_n \]

converges.

(b) \( 1 < \rho \leq \infty \). For all indices beyond some integer \( M \), we have \( \sqrt[n]{a_n} > 1 \), so that \( a_n > 1 \) for \( n > M \). The terms of the series do not converge to zero. The series diverges by the \( n \)th-Term Test.

(c) \( \rho = 1 \). The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) show that the test is not conclusive when \( \rho = 1 \). The first series diverges and the second converges, but in both cases \( \sqrt[n]{a_n} \to 1 \). \[ \square \]

**EXAMPLE 3** Applying the Root Test

Which of the following series converges, and which diverges?

(a) \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \) \hspace{1cm} (b) \( \sum_{n=1}^{\infty} \frac{2^n}{n^2} \) \hspace{1cm} (c) \( \sum_{n=1}^{\infty} \left( \frac{1}{1 + n} \right)^n \)

**Solution**

(a) \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \) converges because

\[ \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\sqrt[n]{n}}{2} \to \frac{1}{2} < 1. \]

(b) \( \sum_{n=1}^{\infty} \frac{2^n}{n^2} \) diverges because

\[ \sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\sqrt[n]{(\sqrt[n]{n})^n}} \to \frac{2}{1} > 1. \]

(c) \( \sum_{n=1}^{\infty} \left( \frac{1}{1 + n} \right)^n \) converges because

\[ \sqrt[n]{\left( \frac{1}{1 + n} \right)^n} = \frac{1}{1 + n} \to 0 < 1. \] \[ \square \]

**EXAMPLE 2** Revisited

Let \( a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even} \end{cases} \) Does \( \sum a_n \) converge?

**Solution** We apply the Root Test, finding that

\[ \sqrt[n]{a_n} = \begin{cases} \sqrt[n]{n/2}, & n \text{ odd} \\ 1/2, & n \text{ even} \end{cases} \]

Therefore,

\[ \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{\sqrt{n}}{2}. \]

Since \( \sqrt[n]{n} \to 1 \) (Section 11.1, Theorem 5), we have \( \lim_{n \to \infty} \sqrt[n]{a_n} = 1/2 \) by the Sandwich Theorem. The limit is less than 1, so the series converges by the Root Test. \[ \square \]