In the calculus of functions of a single variable, we used our knowledge of lines to study curves in the plane. We investigated tangents and found that, when highly magnified, differentiable curves were effectively linear.

To study the calculus of functions of more than one variable in the next chapter, we start with planes and use our knowledge of planes to study the surfaces that are the graphs of functions in space.

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

### Lines and Line Segments in Space

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a vector giving the direction of the line.

Suppose that $L$ is a line in space passing through a point parallel to a vector $v$. Then $L$ is the set of all points $(x, y, z)$ for which $P_0$ is parallel to $v$ (Figure 12.35). Thus, for some scalar parameter $t$. The value of $t$ depends on the location of the point $P$ along the line, and the domain of $t$ is $(-\infty, \infty)$. The expanded form of the equation $P_0 \vec{P} = tv$ is

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(v_1i + v_2j + v_3k),$$

which can be rewritten as

$$xi + yj + zk = x_0i + y_0j + z_0k + t(v_1i + v_2j + v_3k). \tag{1}$$

If $r(t)$ is the position vector of a point $P(x, y, z)$ on the line and $r_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

#### Vector Equation for a Line

A vector equation for the line $L$ through $P_0(x_0, y_0, z_0)$ parallel to $v$ is

$$r(t) = r_0 + tv, \quad -\infty < t < \infty, \tag{2}$$

where $r$ is the position vector of a point $P(x, y, z)$ on $L$ and $r_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter $t$:

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$. 

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**FIGURE 12.35** A point $P$ lies on $L$ through $P_0$ parallel to $v$ if and only if $P_0 \vec{P}$ is a scalar multiple of $v$. 

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Parametric Equations for a Line
The standard parametrization of the line through \( P_0(x_0, y_0, z_0) \) parallel to \( v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \) is
\[
\begin{align*}
  x &= x_0 + tv_1, \\
  y &= y_0 + tv_2, \\
  z &= z_0 + tv_3, \\
\end{align*}
\]
\(-\infty < t < \infty \) (3)

**EXAMPLE 1**  Parametrizing a Line Through a Point Parallel to a Vector
Find parametric equations for the line through \((-2, 0, 4)\) parallel to \( v = 2 \mathbf{i} + 4 \mathbf{j} - 2 \mathbf{k} \) (Figure 12.36).

**Solution**  With \( P_0(x_0, y_0, z_0) = (-2, 0, 4) \) and \( v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} = 2 \mathbf{i} + 4 \mathbf{j} - 2 \mathbf{k} \), Equations (3) become
\[
\begin{align*}
  x &= -2 + 2t, \\
  y &= 4t, \\
  z &= 4 - 2t. \\
\end{align*}
\]

**EXAMPLE 2**  Parametrizing a Line Through Two Points
Find parametric equations for the line through \( P(-3, 2, -3) \) and \( Q(1, -1, 4) \).

**Solution**  The vector
\[
\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + ((-1) - 2)\mathbf{j} + (4 - (-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}
\]
is parallel to the line, and Equations (3) with \((x_0, y_0, z_0) = (-3, 2, -3)\) give
\[
\begin{align*}
  x &= -3 + 4t, \\
  y &= 2 - 3t, \\
  z &= -3 + 7t. \\
\end{align*}
\]

We could have chosen \( Q(1, -1, 4) \) as the “base point” and written
\[
\begin{align*}
  x &= 1 + 4t, \\
  y &= -1 - 3t, \\
  z &= 4 + 7t. \\
\end{align*}
\]
These equations serve as well as the first; they simply place you at a different point on the line for a given value of \( t \).

Notice that parametrizations are not unique. Not only can the “base point” change, but so can the parameter. The equations \( x = -3 + 4t^2, y = 2 - 3t^3, \) and \( z = -3 + 7t^5 \) also parametrize the line in Example 2.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the \( t \)-values for the endpoints and restrict \( t \) to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.

**EXAMPLE 3**  Parametrizing a Line Segment
Parametrize the line segment joining the points \( P(-3, 2, -3) \) and \( Q(1, -1, 4) \) (Figure 12.37).

**Solution**  We begin with equations for the line through \( P \) and \( Q \), taking them, in this case, from Example 2:
\[
\begin{align*}
  x &= -3 + 4t, \\
  y &= 2 - 3t, \\
  z &= -3 + 7t. \\
\end{align*}
\]
We observe that the point
\[(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)\]
on the line passes through \(P(-3, 2, -3)\) at \(t = 0\) and \(Q(1, -1, 4)\) at \(t = 1\). We add the restriction \(0 \leq t \leq 1\) to parametrize the segment:
\[x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1.\]

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position \(P_0(x_0, y_0, z_0)\) and moving in the direction of vector \(v\). Rewriting Equation (2), we have
\[r(t) = r_0 + tv\]
\[= r_0 + t|v| \frac{v}{|v|}.\]  
(4)

In other words, the position of the particle at time \(t\) is its initial position plus its distance moved (speed \(\times\) time) in the direction \(v/|v|\) of its straight-line motion.

**EXAMPLE 4**  Flight of a Helicopter

A helicopter is to fly directly from a helipad at the origin in the direction of the point \((1, 1, 1)\) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

**Solution**  We place the origin at the starting position (helipad) of the helicopter. Then the unit vector
\[
\mathbf{u} = \frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k}
\]
gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time \(t\) is
\[
r(t) = r_0 + t\text{(speed)}\mathbf{u}
\]
\[
= 0 + t(60) \left( \frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k} \right)
\]
\[
= 20\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}).
\]

When \(t = 10\) sec,
\[
r(10) = 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})
\]
\[
= \left( 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \right).
\]

After 10 sec of flight from the origin toward \((1, 1, 1)\), the helicopter is located at the point \((200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})\) in space. It has traveled a distance of (60 ft/sec)(10 sec) = 600 ft, which is the length of the vector \(r(10)\).
The Distance from a Point to a Line in Space

To find the distance from a point \( S \) to a line that passes through a point \( P \) parallel to a vector \( v \), we find the absolute value of the scalar component of \( \overrightarrow{PS} \) in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is, which is

\[
| \overrightarrow{PS} | \sin \theta,
\]

where \( \theta \) is the angle between \( \overrightarrow{PS} \) and \( v \).

**Distance from a Point \( S \) to a Line Through \( P \) Parallel to \( v \)**

\[
d = \frac{| \overrightarrow{PS} \times v |}{|v|} \hspace{1cm} (5)
\]

**EXAMPLE 5** Finding Distance from a Point to a Line

Find the distance from the point \( S(1, 1, 5) \) to the line

\[
L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.
\]

**Solution** We see from the equations for \( L \) that \( L \) passes through \( P(1, 3, 0) \) parallel to \( v = i - j + 2k \). With

\[
\overrightarrow{PS} = (1 - 1)i + (1 - 3)j + (5 - 0)k = -2j + 5k
\]

and

\[
\overrightarrow{PS} \times v = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i + 5j + 2k,
\]

Equation (5) gives

\[
d = \frac{|\overrightarrow{PS} \times v|}{|v|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.
\]

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane \( M \) passes through a point \( P_0(x_0, y_0, z_0) \) and is normal to the nonzero vector \( n = Ai + Bj + Ck \). Then \( M \) is the set of all points \( P(x, y, z) \) for which \( \overrightarrow{P_0P} \) is orthogonal to \( n \) (Figure 12.39). Thus, the dot product \( n \cdot \overrightarrow{P_0P} = 0 \). This equation is equivalent to

\[
(Ai + Bj + Ck) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0
\]

or

\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.
\]
EXAMPLE 6 Finding an Equation for a Plane

Find an equation for the plane through \( P_0(-3, 0, 7) \) perpendicular to \( \mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

Solution The component equation is

\[
5(x + 3) + 2(y - 0) + (-1)(z - 7) = 0.
\]

Simplifying, we obtain

\[
5x + 15 + 2y - z + 7 = 0
\]

\[
5x + 2y - z = -22.
\]

Notice in Example 6 how the components of \( \mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k} \) became the coefficients of \( x, y, \) and \( z \) in the equation \( 5x + 2y - z = -22 \). The vector \( \mathbf{n} = Ai + Bj + Ck \) is normal to the plane \( Ax + By + Cz = D \).

EXAMPLE 7 Finding an Equation for a Plane Through Three Points

Find an equation for the plane through \( A(0, 0, 1), B(2, 0, 0), \) and \( C(0, 3, 0) \).

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 0 & -1 \\
0 & 3 & -1
\end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}
\]

is normal to the plane. We substitute the components of this vector and the coordinates of \( A(0, 0, 1) \) into the component form of the equation to obtain

\[
3(x - 0) + 2(y - 0) + 6(z - 1) = 0
\]

\[
3x + 2y + 6z = 6.
\]

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are parallel if and only if their normals are parallel, or \( \mathbf{n}_1 = k\mathbf{n}_2 \) for some scalar \( k \). Two planes that are not parallel intersect in a line.
12.5 Lines and Planes in Space

EXAMPLE 8  Finding a Vector Parallel to the Line of Intersection of Two Planes

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

**Solution**  The line of intersection of two planes is perpendicular to both planes’ normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes’ line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$  

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well.

EXAMPLE 9  Parametrizing the Line of Intersection of Two Planes

Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

**Solution**  We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for $x$ and $y$ simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$  

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for $y$ and $z$. The different choices would simply give different parametrizations of the same line.

Sometimes we want to know where a line and a plane intersect. For example, if we are looking at a flat plate and a line segment passes through it, we may be interested in knowing what portion of the line segment is hidden from our view by the plate. This application is used in computer graphics (Exercise 74).

EXAMPLE 10  Finding the Intersection of a Line and a Plane

Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

**Solution**  The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$
lies in the plane if its coordinates satisfy the equation of the plane, that is, if
\[
3 \left( \frac{8}{3} + 2t \right) + 2(-2t) + 6(1 + t) = 6
\]
\[
8 + 6t - 4t + 6 + 6t = 6
\]
\[
8t = -8
\]
\[
t = -1.
\]
The point of intersection is
\[
(t, y, z) = \left( \frac{8}{3} - 2, 2, 1 - 1 \right) = \left( \frac{2}{3}, 2, 0 \right).
\]

**The Distance from a Point to a Plane**

If \( P \) is a point on a plane with normal \( \mathbf{n} \), then the distance from any point \( S \) to the plane is the length of the vector projection of \( \overrightarrow{PS} \) onto \( \mathbf{n} \). That is, the distance from \( S \) to the plane is
\[
d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{||\mathbf{n}||} \right| \tag{6}
\]
where \( \mathbf{n} = Ai + Bj + Ck \) is normal to the plane.

**EXAMPLE 11** Finding the Distance from a Point to a Plane

Find the distance from \( S(1, 1, 3) \) to the plane \( 3x + 2y + 6z = 6 \).

**Solution** We find a point \( P \) in the plane and calculate the length of the vector projection of \( \overrightarrow{PS} \) onto a vector \( \mathbf{n} \) normal to the plane (Figure 12.41). The coefficients in the equation \( 3x + 2y + 6z = 6 \) give
\[
\mathbf{n} = 3i + 2j + 6k.
\]
The points on the plane easiest to find from the plane’s equation are the intercepts. If we take \( P \) to be the \( y \)-intercept \((0, 3, 0)\), then

\[
\vec{PS} = (1 - 0)i + (1 - 3)j + (3 - 0)k = i - 2j + 3k,
\]

\[
|n| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.
\]

The distance from \( S \) to the plane is

\[
d = \frac{|\vec{PS} \cdot n|}{|n|}
\]

\[
= \left| (i - 2j + 3k) \cdot \left( \frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k \right) \right|
\]

\[
= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}.
\]

**Angles Between Planes**

The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors (Figure 12.42).

**EXAMPLE 12** Find the angle between the planes \( 3x - 6y - 2z = 15 \) and \( 2x + y - 2z = 5 \).

**Solution** The vectors

\[
\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}
\]

are normals to the planes. The angle between them is

\[
\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right)
\]

\[
= \cos^{-1} \left( \frac{4}{21} \right)
\]

\[
\approx 1.38 \text{ radians. About } 79 \text{ deg}
\]