1. **Submarine hunting** Two surface ships on maneuvers are trying to determine a submarine’s course and speed to prepare for an aircraft intercept. As shown here, ship A is located at (4, 0, 0), whereas ship B is located at (0, 5, 0). All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $\mathbf{s} = 2\mathbf{i} + 3\mathbf{j} - (1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i} - 6\mathbf{j} - \mathbf{k}$. Four minutes ago, the submarine was located at $(2, -1, -1/3)$. The aircraft is due in 20 min. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?

2. **A helicopter rescue** Two helicopters, $H_1$ and $H_2$, are traveling together. At time $t = 0$, they separate and follow different straight-line paths given by

   $$H_1: \quad x = 6 + 40t, \quad y = -3 + 10t, \quad z = -3 + 2t$$

   $$H_2: \quad x = 6 + 110t, \quad y = -3 + 4t, \quad z = -3 + t.$$ 

   Time $t$ is measured in hours and all coordinates are measured in miles. Due to system malfunctions, $H_2$ stops its flight at $(446, 13, 1)$ and, in a negligible amount of time, lands at $(446, 13, 0)$. Two hours later, $H_1$ is advised of this fact and heads toward $H_2$ at 150 mph. How long will it take $H_1$ to reach $H_2$?

3. **Torque** The operator’s manual for the Toro® 21 in. lawnmower says “tighten the spark plug to 15 ft-lb (20.4 N·m).” If you are installing the plug with a 10.5-in. socket wrench that places the center of your hand 9 in. from the axis of the spark plug, about how hard should you pull? Answer in pounds.
4. Rotating body  The line through the origin and the point \( A(1, 1, 1) \) is the axis of rotation of a right body rotating with a constant angular speed of \( \frac{3}{2} \) rad/sec. The rotation appears to be clockwise when we look toward the origin from \( A \). Find the velocity \( v \) of the point of the body that is at the position \( B(1, 3, 2) \).

5. Determinants and planes
   a. Show that
   \[
   \begin{vmatrix}
   x_1 - x & y_1 - y & z_1 - z \\
   x_2 - x & y_2 - y & z_2 - z \\
   x_3 - x & y_3 - y & z_3 - z
   \end{vmatrix} = 0
   \]
   is an equation for the plane through the three noncollinear points \( P(x_1, y_1, z_1) \), \( P(x_2, y_2, z_2) \), and \( P(x_3, y_3, z_3) \).
   b. What set of points in space is described by the equation
   \[
   \begin{vmatrix}
   x & y & z & 1 \\
   a_1 & b_1 & c_1 & 1 \\
   a_2 & b_2 & c_2 & 1 \\
   a_3 & b_3 & c_3 & 1
   \end{vmatrix} = 0?
   \]

6. Determinants and lines  Show that the lines
   \[
   x = a_1 s + b_1 t, y = a_2 s + b_2 t, z = a_3 s + b_3, -\infty < s < \infty,
   \]
   and
   \[
   x = c_1 t + d_1, y = c_2 t + d_2, z = c_3 t + d_3, -\infty < t < \infty,
   \]
   intersect or are parallel if and only if
   \[
   \begin{vmatrix}
   a_1 & c_1 & b_1 - d_1 \\
   a_2 & c_2 & b_2 - d_2 \\
   a_3 & c_3 & b_3 - d_3
   \end{vmatrix} = 0.
   \]

7. Parallelogram  The accompanying figure shows parallelogram \( ABCD \) and the midpoint \( P \) of diagonal \( BD \).
   a. Express \( \overrightarrow{BD} \) in terms of \( \overrightarrow{AB} \) and \( \overrightarrow{AD} \).
   b. Express \( \overrightarrow{AP} \) in terms of \( \overrightarrow{AB} \) and \( \overrightarrow{AD} \).
   c. Prove that \( P \) is also the midpoint of diagonal \( AC \).

8. In the figure here, \( D \) is the midpoint of side \( AB \) of triangle \( ABC \), and \( E \) is one-third of the way between \( C \) and \( B \). Use vectors to prove that \( F \) is the midpoint of line segment \( CD \).

9. Use vectors to show that the distance from \( P(x_1, y_1) \) to the line \( ax + by = c \) is
   \[
   d = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.
   \]

10. Use vectors to show that the distance from \( P(x_1, y_1, z_1) \) to the plane \( Ax + By + Cz = D \) is
    \[
    d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.
    \]

11. a. Show that the distance between the parallel planes \( Ax + By + Cz = D_1 \) and \( Ax + By + Cz = D_2 \) is
    \[
    d = \frac{|D_1 - D_2|}{|A i + B j + C k|}.
    \]
   b. Find the distance between the planes \( 2x + 3y - z = 6 \) and \( 2x + 3y - z = 12 \).
   c. Find an equation for the plane parallel to the plane \( 2x - y + z = -4 \) if the point \( (3, 2, -1) \) is equidistant from the two planes.
   d. Write equations for the planes that lie parallel to and 5 units away from the plane \( x - 2y + z = 3 \).

12. Prove that four points \( A, B, C, \) and \( D \) are coplanar (lie in a common plane) if and only if \( \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0 \).

13. The projection of a vector on a plane   Let \( P \) be a plane in space and let \( v \) be a vector. The vector projection of \( v \) onto the plane \( P \), \( \text{proj}_P \mathbf{v} \), can be defined informally as follows. Suppose the sun is shining so that its rays are normal to the plane \( P \). Then \( \text{proj}_P \mathbf{v} \) is the "shadow" of \( \mathbf{v} \) onto \( P \). If \( P \) is the plane \( x + 2y + 6z = 6 \) and \( \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k} \), find \( \text{proj}_P \mathbf{v} \).

14. The accompanying figure shows nonzero vectors \( \mathbf{v}, \mathbf{w}, \) and \( \mathbf{z} \), with \( \mathbf{z} \) orthogonal to the line \( L \), and \( \mathbf{v} \) and \( \mathbf{w} \) making equal angles \( \beta \) with \( L \). Assuming \( |\mathbf{v}| = |\mathbf{w}| \), find \( \mathbf{w} \) in terms of \( \mathbf{v} \) and \( \mathbf{z} \).
In physics, the law of gravitation

25. Point masses and gravitation

Show that dot multiplication is positive definite

Show that and are orthogonal.

Show that for any vectors

17. Cross and dot products

By forming the cross product of two appropriate vectors, derive

where \( \mathbf{r} \) is the vector from \( P \) to \( Q \) and \( G \) is the universal gravitational constant. Moreover, if \( Q_1, \ldots, Q_n \) are (point) masses with mass \( m_1, \ldots, m_n \), respectively, then the force on \( P \) due to all the \( Q_i \)'s is

\[
\mathbf{F} = \sum_{i=1}^{n} \frac{Gm_i m}{|\mathbf{r}_i|^3} \mathbf{r}_i
\]

where \( \mathbf{r}_i \) is the vector from \( P \) to \( Q_i \).

\[
\begin{array}{c}
\begin{array}{c}
Q_n \quad Q_{n-1} \quad Q_{n-2} \quad Q_1 \quad Q_0 \\
-nd \quad \ldots \quad -2d \quad -d \quad 0 \quad d \quad 2d \quad \ldots \quad nd
\end{array}
\end{array}
\]

a. Let point \( P \) with mass \( M \) be located at the point \((0, d)\), \( d > 0 \), in the coordinate plane. For \( i = -n, -n + 1, \ldots, -1, 0, 1, \ldots, n \), let \( Q_i \) be located at the point \((id, 0)\) and have mass \( m_i \). Find the magnitude of the gravitational force on \( P \) due to all the \( Q_i \)'s.

b. Is the limit as \( n \to \infty \) of the magnitude of the force on \( P \) finite? Why, or why not?

26. Relativistic sums

Einstein’s special theory of relativity roughly says that with respect to a reference frame (coordinate system) no material object can travel as fast as \( c \), the speed of light. So, if \( \ddot{x} \) and \( \ddot{y} \) are two velocities such that \( |\ddot{x}| < c \) and \( |\ddot{y}| < c \), then the relativistic sum \( \ddot{x} \oplus \ddot{y} \) of \( \ddot{x} \) and \( \ddot{y} \) must have length less than \( c \). Einstein’s special theory of relativity says that

\[
\ddot{x} \oplus \ddot{y} = \frac{\ddot{x} + \ddot{y}}{1 + \frac{\ddot{x} \cdot \ddot{y}}{c^2}} + \frac{1}{\gamma} \frac{\ddot{x} \times (\ddot{x} \times \ddot{y})}{1 + \frac{\ddot{x} \cdot \ddot{y}}{c^2}}
\]

where

\[
\gamma = \frac{1}{\sqrt{1 - \frac{\ddot{x} \cdot \ddot{x}}{c^2}}}
\]

It can be shown that if \( |\ddot{x}| < c \) and \( |\ddot{y}| < c \), then \( |\ddot{x} \oplus \ddot{y}| < c \).

This exercise deals with two special cases.

a. Prove that if \( \ddot{x} \) and \( \ddot{y} \) are orthogonal, \( |\ddot{x}| < c \), \( |\ddot{y}| < c \), then \( |\ddot{x} \oplus \ddot{y}| < c \).

b. Prove that if \( \ddot{x} \) and \( \ddot{y} \) are parallel, \( |\ddot{x}| < c \), \( |\ddot{y}| < c \), then \( |\ddot{x} \oplus \ddot{y}| < c \).

c. Compute \( \lim_{c \to \infty} \ddot{x} \oplus \ddot{y} \).