13.5 Torsion and the Unit Binormal Vector \( \mathbf{B} \)

### EXERCISES 13.5

**Finding Torsion and the Binormal Vector**

For Exercises 1–8 you found \( \mathbf{T}, \mathbf{N}, \) and \( \kappa \) in Section 13.4 (Exercises 9–16). Find now \( \mathbf{B} \) and for these space curves.

1. \( \mathbf{r}(t) = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k} \)
2. \( \mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3\mathbf{k} \)
3. \( \mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2\mathbf{k} \)
4. \( \mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \)
5. \( \mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j} \), \( t > 0 \)
6. \( \mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j} \), \( 0 < t < \pi/2 \)
7. \( \mathbf{r}(t) = t\mathbf{i} + (a \cosh (t/a))\mathbf{j} \), \( a > 0 \)
8. \( \mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + tk \)

**Tangential and Normal Components of Acceleration**

In Exercises 9 and 10, write \( \mathbf{a} \) in the form \( a_T \mathbf{T} + a_N \mathbf{N} \) without finding \( \mathbf{T} \) and \( \mathbf{N} \).

9. \( \mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k} \)
10. \( \mathbf{r}(t) = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k} \)

In Exercises 11–14, write \( \mathbf{a} \) in the form \( a_T \mathbf{T} + a_N \mathbf{N} \) at the given value of \( t \) without finding \( \mathbf{T} \) and \( \mathbf{N} \).

11. \( \mathbf{r}(t) = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \), \( t = 1 \)
12. \( \mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t^2\mathbf{k} \), \( t = 0 \)
13. \( \mathbf{r}(t) = t^2\mathbf{i} + (t + (1/3)t^3)\mathbf{j} + (t - (1/3)t^3)\mathbf{k} \), \( t = 0 \)
14. \( \mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \), \( t = 0 \)

In Exercises 15 and 16, find \( \mathbf{r}, \mathbf{T}, \mathbf{N}, \) and \( \mathbf{B} \) at the given value of \( t \). Then find equations for the osculating, normal, and rectifying planes at that value of \( t \).

15. \( \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \), \( t = \pi/4 \)
16. \( \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \), \( t = 0 \)

**Physical Applications**

17. The speedometer on your car reads a steady 35 mph. Could you be accelerating? Explain.
18. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.
19. Can anything be said about the speed of a particle whose acceleration is always orthogonal to its velocity? Give reasons for your answer.
20. An object of mass \( m \) travels along the parabola \( y = x^2 \) with a constant speed of 10 units/sec. What is the force on the object due to its acceleration at \((0, 0)\) at \((1/2, 2)\)? Write your answers in terms of \( i \) and \( j \). (Remember Newton's law, \( \mathbf{F} = ma \).)

21. The following is a quotation from an article in The American Mathematical Monthly, titled “Curvature in the Eighties” by Robert Osserman (October 1990, page 731):

Curvature also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectory.

Explain mathematically why the second sentence of the quotation is true.

22. Show that a moving particle will move in a straight line if the normal component of its acceleration is zero.

23. A sometime shortcut to curvature If you already know \(|a_N|\) and \(|\mathbf{v}|\), then the formula \( a_N = \kappa |\mathbf{v}|^2 \) gives a convenient way to find the curvature. Use this to find the curvature and radius of curvature of the curve

\[
\mathbf{r}(t) = (\cos t + t \sin t)i + (\sin t - t \cos t)j, \quad t > 0.
\]

(Take \(a_N\) and \(|\mathbf{v}|\) from Example 1.)

24. Show that \( \kappa \) and \( \tau \) are both zero for the line

\[
\mathbf{r}(t) = (x_0 + At)i + (y_0 + Bt)j + (z_0 + Ct)k.
\]

25. What can be said about the torsion of a smooth plane curve \( \mathbf{r}(t) = f(t)i + g(t)j \)? Give reasons for your answer.

26. The torsion of a helix In Example 2, we found the torsion of the helix

\[
\mathbf{r}(t) = (a \cos t)i + (a \sin t)j + btk, \quad a, b \neq 0
\]

to be \( \tau = b/(a^2 + b^2)\). What is the largest value \( \tau \) can have for a given value of \( a \)? Give reasons for your answer.

27. Differentiable curves with zero torsion lie in planes That a sufficiently differentiable curve with zero torsion lies in a plane is a special case of the fact that a particle whose velocity remains perpendicular to a fixed vector \( \mathbf{C} \) moves in a plane perpendicular to \( \mathbf{C} \). This, in turn, can be viewed as the solution of the following problem in calculus.

Suppose \( \mathbf{r}(t) = f(t)i + g(t)j + h(t)k \) is twice differentiable for all \( t \) in an interval \([a, b]\), that \( \mathbf{r} = 0 \) when \( t = a \), and that \( \mathbf{v} \cdot \mathbf{k} = 0 \) for all \( t \) in \([a, b]\). Then \( h(t) = 0 \) for all \( t \) in \([a, b]\).

Solve this problem. (Hint: Start with \( \mathbf{a} = d^2\mathbf{r}/dt^2 \) and apply the initial conditions in reverse order.)

28. A formula that calculates \( \tau \) from \( \mathbf{B} \) and \( \mathbf{v} \) If we start with the definition \( \tau = -(d\mathbf{B}/ds) \cdot \mathbf{N} \) and apply the Chain Rule to rewrite \( d\mathbf{B}/ds \) as

\[
\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt} \frac{dt}{ds} = \frac{d\mathbf{B}}{dt} \frac{1}{|\mathbf{v}|},
\]

we arrive at the formula

\[
\tau = -\frac{1}{|\mathbf{v}|} \left( \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right).
\]

The advantage of this formula over Equation (6) is that it is easier to derive and state. The disadvantage is that it can take a lot of work to evaluate without a computer. Use the new formula to find the torsion of the helix in Example 2.

**COMPUTER EXPLORATIONS**

**Curvature, Torsion, and the TNB Frame**

Rounding the answers to four decimal places, use a CAS to find \( \mathbf{v}, \mathbf{a}, \mathbf{N}, \mathbf{B}, \kappa, \tau \), and the tangential and normal components of acceleration for the curves in Exercises 29–32 at the given values of \( t \).

29. \( \mathbf{r}(t) = (t \cos t)i + (t \sin t)j + tk, \quad t = \sqrt{3} \)

30. \( \mathbf{r}(t) = (e^t \cos t)i + (e^t \sin t)j + e^t \mathbf{k}, \quad t = \ln 2 \)

31. \( \mathbf{r}(t) = (t - \sin t)i + (1 - \cos t)j + \sqrt{2}tk, \quad t = -3\pi \)

32. \( \mathbf{r}(t) = (3t - t^3)i + (3r^2)j + (3t + r^3)k, \quad t = 1 \)