EXERCISES 14.4

Chain Rule: One Independent Variable

In Exercises 1–6, (a) express \( dw/dt \) as a function of \( t \), both by using the Chain Rule and by expressing \( w \) in terms of \( t \) and differentiating directly with respect to \( t \). Then (b) evaluate \( dw/dt \) at the given value of \( t \).

1. \( w = x^2 + y^2, \ x = \cos t, \ y = \sin t; \ t = \pi \)
2. \( w = x^2 + y^2, \ x = \cos t + \sin t, \ y = \cos t - \sin t; \ t = 0 \)
3. \( w = \frac{x}{y} + \frac{y}{x}, \ x = \cos^2 t, \ y = \sin^2 t; \ z = 1/t; \ t = 3 \)
4. \( w = \ln (x^2 + y^2 + z^2), \ x = \cos t, \ y = \sin t, \ z = 4\sqrt{t}; \ t = 3 \)
5. \( w = 2ye^x - \sin z, \ x = \ln (t^2 + 1), \ y = \tan^{-1} t, \ z = e^t; \ t = 1 \)
6. \( w = z - \sin xy, \ x = t, \ y = \ln t, \ z = e^{t-1}; \ t = 1 \)

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, (a) express \( \partial z/\partial u \) and \( \partial z/\partial v \) as functions of \( u \) and \( v \) both by using the Chain Rule and by expressing \( z \) directly in terms of \( u \) and \( v \) before differentiating. Then (b) evaluate \( \partial z/\partial u \) and \( \partial z/\partial v \) at the given point \((u, v)\).

7. \( z = 4e^t \ln y, \ x = \ln (u \cos v), \ y = u \sin v; \) \((u, v) = (2, \pi/4)\)
8. \( z = \tan^{-1} (x/y), \ x = u \cos v, \ y = u \sin v; \) \((u, v) = (1.3, \pi/6)\)

In Exercises 9 and 10, (a) express \( \partial w/\partial u \) and \( \partial w/\partial v \) as functions of \( u \) and \( v \) both by using the Chain Rule and by expressing \( w \) directly in terms of \( u \) and \( v \) before differentiating. Then (b) evaluate \( \partial w/\partial u \) and \( \partial w/\partial v \) at the given point \((u, v)\).

9. \( w = xy + yz + xz, \ x = u + v, \ y = u - v, \ z = uv; \) \((u, v) = (1/2, 1)\)
10. \( w = \ln (x^2 + y^2 + z^2), \ x = ue^v \sin u, \ y = ue^v \cos u, \ z = ue^v; \) \((u, v) = (-2, 0)\)

In Exercises 11 and 12, (a) express \( \partial u/\partial x, \partial u/\partial y, \) and \( \partial u/\partial z \) as functions of \( x, y, \) and \( z \) both by using the Chain Rule and by expressing \( u \) directly in terms of \( x, y, \) and \( z \) before differentiating. Then (b) evaluate \( \partial u/\partial x, \partial u/\partial y, \) and \( \partial u/\partial z \) at the given point \((x, y, z)\).

11. \( u = \frac{p - q}{q - r}, \ p = x + y + z, \ q = x - y + z, \ r = x + y - z; \) \((x, y, z) = (\sqrt{3}, 2, 1)\)
12. \( u = e^{q \sin^{-1} p}, \ p = \sin x, \ q = z^2 \ln y, \ r = 1/z; \) \((x, y, z) = (\pi/4, 1/2, -1/2)\)

Using a Tree Diagram

In Exercises 13–24, draw a tree diagram and write a Chain Rule formula for each derivative.

13. \( \frac{dz}{dt} \) for \( z = f(x, y), \ x = g(t), \ y = h(t) \)
14. \( \frac{dz}{dt} \) for \( z = f(u, v, w), \ u = g(t), \ v = h(t), \ w = k(t) \)
15. \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \) for \( w = h(x, y, z), \ x = f(u, v), \ y = g(u, v), \ z = k(u, v) \)
Finding Specified Partial Derivatives

33. Find $\frac{\partial w}{\partial r}$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s, y = \cos (r + s), z = \sin (r + s)$.

34. Find $\frac{\partial w}{\partial v}$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u, y = u + v, z = \cos u$.

35. Find $\frac{\partial w}{\partial v}$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1, y = 2u + v - 2$.

36. Find $\frac{\partial z}{\partial u}$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y$, $x = u^2 + v^2, y = uv$.

37. Find $\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}$ when $u = \ln 2, v = 1$ if $z = 5 \tan^{-1} x$ and $x = e^u + \ln v$.

38. Find $\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}$ when $u = 1$ and $v = -2$ if $z = \ln q$ and $q = \sqrt{u + v^3} \tan^{-1} u$.

Theory and Examples

39. Changing voltage in a circuit The voltage $V$ in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance $R$ is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.

40. Changing dimensions in a box The lengths $a, b, \text{ and } c$ of the edges of a rectangular box are changing with time. At the instant in question, $a = 1 \text{ m, } b = 2 \text{ m, } c = 3 \text{ m, } da/dt = db/dt = 1 \text{ m/sec,}$

t and $dc/dt = -3 \text{ m/sec.}$ At what rates are the box’s volume $V$ and surface area $S$ changing at that instant? Are the box’s interior diagonals increasing in length or decreasing?

41. If $f(u, v, w)$ is differentiable and $u = x - y, v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$
a. Show that
\[
\frac{\partial w}{\partial r} = f_x \cos \theta + f_y \sin \theta
\]
and
\[
\frac{1}{r} \frac{\partial w}{\partial \theta} = -f_x \sin \theta + f_y \cos \theta.
\]
b. Solve the equations in part (a) to express \(f_x\) and \(f_y\) in terms of \(\partial w/\partial r\) and \(\partial w/\partial \theta\).
c. Show that
\[
(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2.
\]

**43. Laplace equations**

Show that if \(w = f(u, v)\) satisfies the Laplace equation \(f_{uu} + f_{vv} = 0\) and if \(u = (x^2 - y^2)/2\) and \(v = xy\), then \(w\) satisfies the Laplace equation \(w_{xx} + w_{yy} = 0\).

**44. Laplace equations**

Let \(w = f(u) + g(v)\), where \(u = x + iy\) and \(v = x - iy\) and \(i = \sqrt{-1}\). Show that \(w\) satisfies the Laplace equation \(w_{xx} + w_{yy} = 0\) if all the necessary functions are differentiable.

**Changes in Functions Along Curves**

**45. Extreme values on a helix**

Suppose that the partial derivatives of a function \(f(x, y, z)\) at points on the helix \(x = \cos t, y = \sin t, z = t\) are
\[
f_x = \cos t, \quad f_y = \sin t, \quad f_z = t^2 + 1 - 2.
\]
At what points on the curve, if any, can \(f\) take on extreme values?

**46. A space curve**

Let \(w = x^2 e^{3z}\). Find the value of \(dw/dt\) at the point \((1, \ln 2, 0)\) on the curve \(x = \cos t, y = \ln (t + 2), z = t\).

**47. Temperature on a circle**

Let \(T = f(x, y)\) be the temperature at the point \((x, y)\) on the circle \(x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\) and suppose that
\[
\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x.
\]
a. Find where the maximum and minimum temperatures on the circle occur by examining the derivatives \(dT/dt\) and \(d^2T/dt^2\).

b. Suppose that \(T = 4x^2 - 4xy + 4y^2\). Find the maximum and minimum values of \(T\) on the circle.

**48. Temperature on an ellipse**

Let \(T = g(x, y)\) be the temperature at the point \((x, y)\) on the ellipse
\[
x = 2\sqrt{2} \cos t, \quad y = \sqrt{2} \sin t, \quad 0 \leq t \leq 2\pi,
\]
and suppose that
\[
\begin{align*}
\frac{\partial T}{\partial x} &= y, \\
\frac{\partial T}{\partial y} &= x.
\end{align*}
\]
a. Locate the maximum and minimum temperatures on the ellipse by examining \(dT/dt\) and \(d^2T/dt^2\).

b. Suppose that \(T = xy - 2\). Find the maximum and minimum values of \(T\) on the ellipse.

**Differentiating Integrals**

Under mild continuity restrictions, it is true that if
\[
F(x) = \int_a^b g(t, x) \, dt,
\]
then \(F'(x) = \int_a^b g_x(t, x) \, dt\). Using this fact and the Chain Rule, we can find the derivative of
\[
F(x) = \int_a^b g(t, x) \, dt
\]
by letting
\[
G(u, x) = \int_a^u g(t, x) \, dt,
\]
where \(u = f(x)\). Find the derivatives of the functions in Exercises 49 and 50.

**49.** \(F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} \, dt\)

**50.** \(F(x) = \int_0^x \sqrt{i^3 + x^2} \, dt\)