Area by Double Integration
In Exercises 1–8, sketch the region bounded by the given lines and curves. Then express the region’s area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line \( x + y = 2 \)
2. The lines \( x = 0, y = 2x \), and \( y = 4 \)
3. The parabola \( x = -y^2 \) and the line \( y = x + 2 \)
4. The parabola \( x = y - y^2 \) and the line \( y = -x \)
5. The curve \( y = e^x \) and the lines \( y = 0, x = 0, \) and \( x = \ln 2 \)
6. The curves \( y = \ln x \) and \( y = 2 \ln x \) and the line \( x = e \), in the first quadrant
7. The parabolas \( x = y^2 \) and \( x = 2y - y^2 \)
8. The parabolas \( x = y^2 - 1 \) and \( x = 2y^2 - 2 \)

Identifying the Region of Integration
The integrals and sums of integrals in Exercises 9–14 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

9. \( \int_0^1 \int_{y^3/3}^{2y} dx \, dy \)
10. \( \int_0^3 \int_{x-(2-x)}^{x} dy \, dx \)
11. \( \int_{\pi/4}^{\pi} \int_{\sin x}^{x} dy \, dx \)
12. \( \int_0^2 \int_{y^2}^{y+2} dx \, dy \)
13. \( \int_{-1}^{1} \int_{x-1}^1 dy \, dx + \int_{0}^{2} \int_{-x/2}^{1-x} dy \, dx \)
14. \( \int_1^4 \int_{\sqrt{x}}^{x^2-4} dy \, dx + \int_0^4 \int_{\sqrt{y} - 2}^{\sqrt{y}} dy \, dx \)

Average Values
15. Find the average value of \( f(x, y) = \sin(x + y) \) over
   a. the rectangle \( 0 \leq x \leq \pi, \ 0 \leq y \leq \pi \)
   b. the rectangle \( 0 \leq x \leq \pi, \ 0 \leq y \leq \pi/2 \)
16. Which do you think will be larger, the average value of \( f(x, y) = xy \) over the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), or the average value of \( f \) over the quarter circle \( x^2 + y^2 \leq 1 \) in the first quadrant? Calculate them to find out.

17. Find the average height of the paraboloid \( z = x^2 + y^2 \) over the square \( 0 \leq x \leq 2, 0 \leq y \leq 2 \).
18. Find the average value of \( f(x, y) = 1/(xy) \) over the square \( \ln 2 \leq x \leq 2 \ln 2, \ln 2 \leq y \leq 2 \ln 2 \).

Constant Density
19. Finding center of mass Find a center of mass of a thin plate of density \( \delta = 3 \) bounded by the lines \( x = 0, y = x, \) and the parabola \( y = 2 - x^2 \) in the first quadrant.
20. Finding moments of inertia and radii of gyration Find the moments of inertia and radii of gyration about the coordinate axes of a thin rectangular plate of constant density \( \delta \) bounded by the lines \( x = 3 \) and \( y = 3 \) in the first quadrant.
21. Finding a centroid Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola \( y^2 = 2x \), and the line \( x + y = 4 \).
22. Finding a centroid Find the centroid of the triangular region cut from the first quadrant by the line \( x + y = 3 \).
23. Finding a centroid Find the centroid of the semicircular region bounded by the x-axis and the curve \( y = \sqrt{1 - x^2} \).
24. Finding a centroid The area of the region in the first quadrant bounded by the parabola \( y = 6x - x^2 \) and the line \( y = x \) is 125/6 square units. Find the centroid.
25. Finding a centroid Find the centroid of the region cut from the first quadrant by the circle \( x^2 + y^2 = a^2 \).
26. Finding a centroid Find the centroid of the region between the x-axis and the arch \( y = \sin x, 0 \leq x \leq \pi \).
27. Finding moments of inertia Find the moment of inertia about the x-axis of a thin plate of density \( \delta = 1 \) bounded by the circle \( x^2 + y^2 = 4 \). Then use your result to find \( I_x \) and \( I_y \) for the plate.
28. Finding a moment of inertia Find the moment of inertia with respect to the y-axis of a thin sheet of constant density \( \delta = 1 \) bounded by the curve \( y = (\sin^2 x)/x^2 \) and the interval \( \pi \leq x \leq 2\pi \) of the x-axis.
29. The centroid of an infinite region Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve \( y = e^x \). (Use improper integrals in the mass-moment formulas.)
30. **The first moment of an infinite plate** Find the first moment about the y-axis of a thin plate of density \( \delta(x, y) = 1 \) covering the infinite region under the curve \( y = e^{-x^2/2} \) in the first quadrant.

31. **Finding a moment of inertia and radius of gyration** Find the moment of inertia and radius of gyration about the x-axis of a thin plate bounded by the parabola \( x = y - y^2 \) and the line \( x + y = 0 \) if \( \delta(x, y) = x + y \).

32. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse \( x^2 + 4y^2 = 12 \) by the parabola \( x = 4y^2 \) if \( \delta(x, y) = 5x \).

33. **Finding a center of mass** Find the center of mass of a thin triangular plate bounded by the y-axis and the lines \( y = x \) and \( y = 2 - x \) if \( \delta(x, y) = 6x + 3y + 3 \).

34. **Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves \( x = y^2 \) and \( x = 2y - y^2 \) if the density at the point \( (x, y) \) is \( \delta(x, y) = y + 1 \).

35. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin rectangular plate cut from the first quadrant by the lines \( x = 6 \) and \( y = 1 \) if \( \delta(x, y) = x + y + 1 \).

36. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the line \( y = 1 \) and the parabola \( y = x^2 \) if the density is \( \delta(x, y) = y + 1 \).

37. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the x-axis, the lines \( x = \pm 1 \), and the parabola \( y = x^2 \) if \( \delta(x, y) = 7y + 1 \).

38. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the x-axis of a thin rectangular plate bounded by the lines \( x = 0, x = 20, y = -1, \) and \( y = 1 \) if \( \delta(x, y) = 1 + (x/20) \).

39. **Center of mass, moments of inertia, and radii of gyration** Find the center of mass, the moment of inertia and radii of gyration about the coordinate axes, and the polar moment of inertia and radius of gyration of a thin triangular plate bounded by the lines \( y = x, y = -x, \) and \( y = 1 \) if \( \delta(x, y) = y + 1 \).

40. **Center of mass, moments of inertia, and radii of gyration** Repeat Exercise 39 for \( \delta(x, y) = 3x^2 + 1 \).

### Variable Density

33. **Finding a moment of inertia and radius of gyration** Find the moment of inertia and radius of gyration about the x-axis of a thin plate bounded by the lines \( x = -y \) and the parabola \( y = x^2 \) if \( \delta(x, y) = e^{-x^2/2} \).

34. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse \( x^2 + 4y^2 = 12 \) by the parabola \( x = 4y^2 \) if \( \delta(x, y) = 5x \).

35. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the curves \( x = y^2 \) and \( x = 2y - y^2 \) if the density at the point \( (x, y) \) is \( \delta(x, y) = y + 1 \).

### Theory and Examples

41. **Bacterium population** If \( f(x, y) = (10,000e^y)/(1 + |x|/2) \) represents the “population density” of a certain bacterium on the \( xy \)-plane, where \( x \) and \( y \) are measured in centimeters, find the total population of bacteria within the rectangle \(-5 \leq x \leq 5 \) and \(-2 \leq y \leq 0 \).

32. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse \( x^2 + 4y^2 = 12 \) by the parabola \( x = 4y^2 \) if \( \delta(x, y) = 5x \).

### Conclusion

42. **Regional population** If \( f(x, y) = 100 \) represents the population density of a planar region on Earth, where \( x \) and \( y \) are measured in miles, find the number of people in the region bounded by the curves \( x = y^2 \) and \( x = 2y - y^2 \).

43. **Appliance design** When we design an appliance, one of the concerns is how hard the appliance will be to tip over. When tipped, it will right itself as long as its center of mass lies on the correct side of the fulcrum, the point on which the appliance is riding as it tips. Suppose that the profile of an appliance of approximately constant density is parabolic, like an old-fashioned radio. It fills the region \( 0 \leq y \leq a(1 - x^2), -1 \leq x \leq 1 \), in the \( xy \)-plane (see accompanying figure). What values of \( a \) will guarantee that the appliance will have to be tipped more than 45° to fall over?

![Diagram of an appliance and its center of mass](image_url)
48. Average temperature in Texas According to the Texas Almanac, Texas has 254 counties and a National Weather Service station in each county. Assume that at time \( t_0 \), each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation to the average temperature in Texas at time \( t_0 \). Your answer should involve information that you would expect to be readily available in the Texas Almanac.

The Parallel Axis Theorem
Let \( L_{c.m.} \) be a line in the \( xy \)-plane that runs through the center of mass of a thin plate of mass \( m \) covering a region in the plane. Let \( L \) be a line in the plane parallel to and \( h \) units away from \( L_{c.m.} \). The Parallel Axis Theorem says that under these conditions the moments of inertia \( I_{c.m.} \) and \( I_L \) of the plate about \( L \) and \( L_{c.m.} \) satisfy the equation

\[
I_L = I_{c.m.} + mh^2.
\]

This equation gives a quick way to calculate one moment when the other moment and the mass are known.

49. Proof of the Parallel Axis Theorem
a. Show that the first moment of a thin flat plate about any line in the plane of the plate through the plate’s center of mass is zero. (Hint: Place the center of mass at the origin with the line along the \( y \)-axis. What does the formula \( \bar{x} = \frac{M_y}{M} \) then tell you?)

b. Use the result in part (a) to derive the Parallel Axis Theorem. Assume that the plane is coordinatized in a way that makes \( L_{c.m.} \) the \( y \)-axis and \( L \) the line \( x = h \). Then expand the integrand of the integral for \( I_L \) to rewrite the integral as the sum of integrals whose values you recognize.

50. Finding moments of inertia
a. Use the Parallel Axis Theorem and the results of Example 4 to find the moments of inertia of the plate in Example 4 about the vertical and horizontal lines through the plate’s center of mass.

b. Use the results in part (a) to find the plate’s moments of inertia about the lines \( x = 1 \) and \( y = 2 \).

Pappus’s Formula
Pappus knew that the centroid of the union of two nonoverlapping plane regions lies on the line segment joining their individual centroids. More specifically, suppose that \( m_1 \) and \( m_2 \) are the masses of thin plates \( P_1 \) and \( P_2 \) that cover nonoverlapping regions in the \( xy \)-plane. Let \( \mathbf{c}_1 \) and \( \mathbf{c}_2 \) be the vectors from the origin to the respective centers of mass of \( P_1 \) and \( P_2 \). Then the center of mass of the union \( P_1 \cup P_2 \) of the two plates is determined by the vector

\[
\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2}{m_1 + m_2}.
\]

Equation (5) is known as Pappus’s formula. For more than two nonoverlapping plates, as long as their number is finite, the formula generalizes to

\[
\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2 + \cdots + m_n \mathbf{c}_n}{m_1 + m_2 + \cdots + m_n}.
\]

This formula is especially useful for finding the centroid of a plate of irregular shape that is made up of pieces of constant density whose centroids we know from geometry. We find the centroid of each piece and apply Equation (6) to find the centroid of the plate.

51. Derive Pappus’s formula (Equation (5)). (Hint: Sketch the plates as regions in the first quadrant and label their centers of mass as \((x_1, y_1)\) and \((x_2, y_2)\). What are the moments of \( P_1 \cup P_2 \) about the coordinate axes?)

52. Use Equation (5) and mathematical induction to show that Equation (6) holds for any positive integer \( n > 2 \).

53. Let \( A, B, \) and \( C \) be the shapes indicated in the accompanying figure. Use Pappus’s formula to find the centroid of

a. \( A \cup B \)

b. \( A \cup C \)

c. \( B \cup C \)

d. \( A \cup B \cup C \).

54. Locating center of mass Locate the center of mass of the carpenter’s square, shown here.

55. An isosceles triangle \( T \) has base \( 2a \) and altitude \( h \). The base lies along the diameter of a semicircular disk \( D \) of radius \( a \) so that the two together make a shape resembling an ice cream cone. What relation must hold between \( a \) and \( h \) to place the centroid of \( T \cup D \) on the common boundary of \( T \) and \( D \)? Inside \( T \)?

56. An isosceles triangle \( T \) of altitude \( h \) has as its base one side of a square \( Q \) whose edges have length \( s \). (The square and triangle do not overlap.) What relation must hold between \( h \) and \( s \) to place the centroid of \( T \cup Q \) on the base of the triangle? Compare your answer with the answer to Exercise 55.