EXERCISES 15.4

Evaluating Triple Integrals in Different Iterations

1. Evaluate the integral in Example 2 taking \( F(x, y, z) = 1 \) to find the volume of the tetrahedron.

2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes \( x = 1, y = 2, \) and \( z = 3 \). Evaluate one of the integrals.

3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane \( z = 8 - x^2 - y^2 \). Evaluate one of the integrals.

4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder \( x^2 + y^2 = 4 \) and the plane \( y = 3 \). Evaluate one of the integrals.

5. **Volume enclosed by paraboloids** Let \( D \) be the region bounded by the paraboloids \( z = 8 - x^2 - y^2 \) and \( z = x^2 + y^2 \). Write six different triple iterated integrals for the volume of \( D \). Evaluate one of the integrals.

6. **Volume inside paraboloid beneath a plane** Let \( D \) be the region bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 2y \). Write triple iterated integrals in the order \( dz \, dx \, dy \) and \( dz \, dy \, dx \) that give the volume of \( D \). Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

1. \( \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) \, dz \, dy \, dx \)

2. \( \int_0^{\sqrt{2}} \int_0^{r^2 - x^2 - y^2} \, dz \, dx \, dy \)

3. \( \int_0^e \int_0^e \int_0^1 \frac{1}{\sqrt[3]{x^2 + y^2 + z^2}} \, dx \, dy \, dz \)

4. \( \int_0^1 \int_0^3 - 3x \int_0^3 - 3x - y \, dz \, dy \)

5. \( \int_0^1 \int_0^1 \int_0^1 (x + y + z) \, dy \, dx \, dz \)

6. \( \int_0^3 \int_0^{\sqrt{9 - x^2}} \int_0^{\sqrt{9 - z^2}} \, dz \, dy \, dx \)

7. \( \int_0^2 \int_0^{\sqrt{4 - y^2}} \int_0^{2x + y} \, dz \, dx \, dy \)

8. \( \int_0^1 \int_0^{2 - x} \int_0^{2 - x - y} \, dz \, dy \, dx \)

9. \( \int_0^1 \int_0^e \int_0^e \ln r \ln s \ln t \, dt \, dr \, dw \) (rsw-space)

10. \( \int_0^e \int_0^e \int_0^e \ln r \ln s \ln t \, dt \, dr \, dw \) (rst-space)

11. \( \int_0^{\pi/4} \int_0^{\ln \sec \theta} \int_0^{2\pi} e^u \, du \, dv \) (tuv-space)

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20. $\int_{0}^{1} \int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \frac{q}{r} \, dq \, dr \, dz$ (pqr-space)

## Volumes Using Triple Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^{1} \int_{0}^{1} \int_{0}^{1-y} dz \, dy \, dx.$$ 

Rewrite the integral as an equivalent iterated integral in the order

a. $dy \, dz \, dx$

b. $dy \, dx \, dz$

c. $dx \, dy \, dz$

d. $dx \, dz \, dy$

e. $dz \, dx \, dy$.

22. Here is the region of integration of the integral

$$\int_{0}^{1} \int_{-1}^{1} \int_{0}^{y^2} dz \, dy \, dx.$$ 

Rewrite the integral as an equivalent iterated integral in the order

a. $dy \, dz \, dx$

b. $dy \, dx \, dz$

c. $dx \, dy \, dz$

d. $dx \, dz \, dy$

e. $dz \, dx \, dy$.

Find the volumes of the regions in Exercises 23–36.

23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$
28. The region in the first octant bounded by the coordinate planes, the plane \( y = 1 - x \), and the surface \( z = \cos(\pi y/2) \), \( 0 \leq x \leq 1 \)

29. The region common to the interiors of the cylinders \( x^2 + y^2 = 1 \) and \( x^2 + z^2 = 1 \), one-eighth of which is shown in the accompanying figure.

30. The region in the first octant bounded by the coordinate planes and the surface \( \sqrt{x^2 + 4z^2} = 16 \)

31. The region in the first octant bounded by the coordinate planes, the plane \( x + y = 4 \), and the cylinder \( x^2 + 4z^2 = 16 \)

32. The region cut from the cylinder \( x^2 + y^2 = 1 \) by the plane \( z = 0 \) and the plane \( x + z = 3 \)

33. The region between the planes \( x + y + 2z = 2 \) and \( 2x + 2y + z = 4 \) in the first octant

34. The finite region bounded by the planes \( z = x, x + z = 8, z = y, y = 8 \), and \( z = 0 \).

35. The region cut from the solid elliptical cylinder \( x^2 + 4y^2 \leq 1 \) by the \( xy \)-plane and the plane \( z = x + 2 \)

36. The region bounded in back by the plane \( x = 0 \), on the front and sides by the parabolic cylinder \( x = 1 - y^2 \), on the top by the paraboloid \( z = x^2 + y^2 \), and on the bottom by the \( xy \)-plane

**Average Values**

In Exercises 37–40, find the average value of \( F(x, y, z) \) over the given region.

37. \( F(x, y, z) = x^2 + 9 \) over the cube in the first octant bounded by the coordinate planes and the planes \( x = 2, y = 2 \), and \( z = 2 \)

38. \( F(x, y, z) = x + y - z \) over the rectangular solid in the first octant bounded by the coordinate planes and the planes \( x = 1, y = 1 \), and \( z = 2 \)

39. \( F(x, y, z) = x^2 + y^2 + z^2 \) over the cube in the first octant bounded by the coordinate planes and the planes \( x = 1, y = 1 \), and \( z = 1 \)

40. \( F(x, y, z) = xyz \) over the cube in the first octant bounded by the coordinate planes and the planes \( x = 2, y = 2 \), and \( z = 2 \)

**Changing the Order of Integration**

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. \( \int_0^2 \int_0^1 \int_0^{4 \cos(\pi z/2)} 2 \sqrt{x} \, dx \, dy \, dz \)

42. \( \int_0^1 \int_0^1 \int_0^{\ln 2} \pi z^2 e^{\pi x} \, dy \, dx \, dz \)

43. \( \int_0^1 \int_0^{\ln 3} \int_0^{\pi y^2} \pi e^{2x} \sin \pi y^2 \, dx \, dy \, dz \)

44. \( \int_0^2 \int_0^{4 - z} \int_0^{2 \sin 2 \pi x} \frac{2 \pi}{4 - z} \, dy \, dz \, dx \)
Theory and Examples

45. Finding upper limit of iterated integral Solve for $a$:
\[ \int_0^1 \int_0^{4-x^2} \int_0^{4-x^2-y} dz \, dy \, dx = \frac{4}{15}. \]

46. Ellipsoid For what value of $c$ is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to $8\pi$?

47. Minimizing a triple integral What domain $D$ in space minimizes the value of the integral
\[ \iiint_D (4x^2 + 4y^2 + z^2 - 4) \, dV? \]
Give reasons for your answer.

48. Maximizing a triple integral What domain $D$ in space maximizes the value of the integral
\[ \iiint_D (1 - x^2 - y^2 - z^2) \, dV? \]
Give reasons for your answer.

COMPUTER EXPLORATIONS

Numerical Evaluations

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

49. $F(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$

50. $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$

51. $F(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$

52. $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \leq 1$