Substitutions in Multiple Integrals

This section shows how to evaluate multiple integrals by substitution. As in single integration, the goal of substitution is to replace complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.3 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region $G$ in the $uv$-plane is transformed one-to-one into the region $R$ in the $xy$-plane by equations of the form

$$x = g(u, v), \quad y = h(u, v),$$

as suggested in Figure 15.47. We call $R$ the image of $G$ under the transformation, and $G$ the preimage of $R$. Any function $f(x, y)$ defined on $R$ can be thought of as a function
15.7 Substitutions in Multiple Integrals

Historical Biography
Carl Gustav Jacob Jacobi
(1804–1851)

The equations $x = g(u, v), y = h(u, v)$ allow us to change an integral over a region $R$ in the $xy$-plane into an integral over a region $G$ in the $uv$-plane. The Jacobian is also denoted by $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$ to help remember how the determinant in Equation (2) is constructed from the partial derivatives of $x$ and $y$. The derivation of Equation (1) is intricate and properly belongs to a course in advanced calculus. We do not give the derivation here.

For polar coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r.
$$

Hence, Equation (1) becomes

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(r \cos \theta, r \sin \theta) |r| \, dr \, d\theta
$$

which is the equation found in Section 15.3.

Figure 15.48 shows how the equations $x = r \cos \theta, y = r \sin \theta$ transform the rectangle $G$: $0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$ into the quarter circle $R$ bounded by $x^2 + y^2 = 1$ in the first quadrant of the $xy$-plane.
Notice that the integral on the right-hand side of Equation (3) is not the integral of \( f(r \cos \theta, r \sin \theta) \) over a region in the polar coordinate plane. It is the integral of the product of \( f(r \cos \theta, r \sin \theta) \) and \( r \) over a region \( G \) in the Cartesian \( r\theta \)-plane.

Here is an example of another substitution.

**EXAMPLE 1** Applying a Transformation to Integrate

Evaluate

\[
\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x - y}{2} \, dx \, dy
\]

by applying the transformation

\[
u = \frac{2x - y}{2}, \quad v = \frac{y}{2}
\]

and integrating over an appropriate region in the \( uv \)-plane.

**Solution** We sketch the region \( R \) of integration in the \( xy \)-plane and identify its boundaries (Figure 15.49).

To apply Equation (1), we need to find the corresponding \( uv \)-region \( G \) and the Jacobian of the transformation. To find them, we first solve Equations (4) for \( x \) and \( y \) in terms of \( u \) and \( v \). Routine algebra gives

\[
x = u + v \quad y = 2v\]  

(5)

We then find the boundaries of \( G \) by substituting these expressions into the equations for the boundaries of \( R \) (Figure 15.49).
The Jacobian of the transformation (again from Equations (5)) is
\[
J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial (u + v)}{\partial u} & \frac{\partial (u + v)}{\partial v} \\ \frac{\partial (2v)}{\partial u} & \frac{\partial (2v)}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.
\]

We now have everything we need to apply Equation (1):
\[
\int_0^4 \int_{x=y/2}^{x=(y/2)+1} 2x - y \frac{dy}{2} dx dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u J(u, v) |du dv
\]
\[
= \int_0^2 \int_0^{1} (u)(2) du dv = \int_0^2 \left[ u^2 \right]_0^1 dv = \int_0^2 dv = 2.
\]

**EXAMPLE 2  Applying a Transformation to Integrate**

Evaluate
\[
\int_0^1 \int_{1-x}^{1-x} \sqrt{x + y (y - 2x)^2} dy dx.
\]

**Solution** We sketch the region \( R \) of integration in the \( xy \)-plane and identify its boundaries (Figure 15.50). The integrand suggests the transformation \( u = x + y \) and \( v = y - 2x \). Routine algebra produces \( x \) and \( y \) as functions of \( u \) and \( v \):
\[
x = \frac{u}{3} - \frac{v}{3}, \quad y = \frac{2u}{3} + \frac{v}{3}
\]
(6)

From Equations (6), we can find the boundaries of the \( uv \)-region \( G \) (Figure 15.50).

<table>
<thead>
<tr>
<th>( xy )-equations for the boundary of ( R )</th>
<th>Corresponding ( uv )-equations for the boundary of ( G )</th>
<th>Simplified ( uv )-equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + y = 1 )</td>
<td>( \left( \frac{u}{3} - \frac{v}{3} \right) + \left( \frac{2u}{3} + \frac{v}{3} \right) = 1 )</td>
<td>( u = 1 )</td>
</tr>
<tr>
<td>( x = 0 )</td>
<td>( \frac{u}{3} - \frac{v}{3} = 0 )</td>
<td>( v = u )</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>( \frac{2u}{3} + \frac{v}{3} = 0 )</td>
<td>( v = -2u )</td>
</tr>
</tbody>
</table>

The Jacobian of the transformation in Equations (6) is
\[
J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.
\]
Applying Equation (1), we evaluate the integral:

\[
\int_0^1 \int_0^{1-x} \sqrt{x + y} (y - 2x)^2 \, dy \, dx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=1} u^{1/2} v^3 |J(u, v)| \, dv \, du
\]

\[
= \int_0^1 \int_{-2u}^{u} u^{1/2} (1/3) v \, dv \, du = \frac{1}{3} \int_0^1 u^{1/2} \left[ \frac{1}{3} v^3 \right]_{v=-2u}^{v=u} \, du
\]

\[
= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3) \, du = \int_0^1 u^{7/2} \, du = \frac{2}{9} u^{9/2} \bigg|_0^1 = \frac{2}{9}.
\]

Substitutions in Triple Integrals

The cylindrical and spherical coordinate substitutions in Section 15.6 are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of three-dimensional regions. The method is like the method for double integrals except that now we work in three dimensions instead of two.

Suppose that a region \(G\) in \(uvw\)-space is transformed one-to-one into the region \(D\) in \(xyz\)-space by differentiable equations of the form

\[
x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w),
\]

as suggested in Figure 15.51. Then any function \(F(x, y, z)\) defined on \(D\) can be thought of as a function

\[
F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)
\]

defined on \(G\). If \(g\), \(h\), and \(k\) have continuous first partial derivatives, then the integral of \(F(x, y, z)\) over \(D\) is related to the integral of \(H(u, v, w)\) over \(G\) by the equation

\[
\iiint_D F(x, y, z) \, dx \, dy \, dz = \iiint_G H(u, v, w) |J(u, v, w)| \, du \, dv \, dw. \tag{7}
\]

**FIGURE 15.51** The equations \(x = g(u, v, w), y = h(u, v, w),\) and \(z = k(u, v, w)\) allow us to change an integral over a region \(D\) in Cartesian \(xyz\)-space into an integral over a region \(G\) in Cartesian \(uvw\)-space.
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The factor \( J(u, v, w) \), whose absolute value appears in this equation, is the **Jacobian determinant**

\[
J(u, v, w) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}
= \frac{\partial (x, y, z)}{\partial (u, v, w)}.
\]

This determinant measures how much the volume near a point in \( G \) is being expanded or contracted by the transformation from \((u, v, w)\) to \((x, y, z)\) coordinates. As in the two-dimensional case, the derivation of the change-of-variable formula in Equation (7) is complicated and we do not go into it here.

For cylindrical coordinates, \( r, \theta, \) and \( z \) take the place of \( u, v, \) and \( w \). The transformation from Cartesian \( r\theta z\)-space to Cartesian \( xyz\)-space is given by the equations

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]

(Figure 15.52). The Jacobian of the transformation is

\[
J(r, \theta, z) = \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta & 0 \\
r \sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.
\]

The corresponding version of Equation (7) is

\[
\iiint_D F(x, y, z) \, dx \, dy \, dz = \iiint_G H(r, \theta, z) |r| \, dr \, d\theta \, dz.
\]

We can drop the absolute value signs whenever \( r \geq 0 \).

For spherical coordinates, \( \rho, \phi, \) and \( \theta \) take the place of \( u, v, \) and \( w \). The transformation from Cartesian \( \rho \theta \phi\)-space to Cartesian \( xyz\)-space is given by

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi
\]

(Figure 15.53). The Jacobian of the transformation is

\[
J(\rho, \phi, \theta) = \begin{vmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{vmatrix} = \rho^2 \sin \phi.
\]
Here is an example of another substitution. Although we could evaluate the integral in this example directly, we have chosen it to illustrate the substitution method in a simple (and fairly intuitive) setting.

**EXAMPLE 3**  
**Applying a Transformation to Integrate**

Evaluate

\[
\iiint_{D} F(x, y, z) \, dx \, dy \, dz = \iiint_{G} H(\rho, \phi, \theta) |p^2 \sin \phi| \, dp \, d\phi \, d\theta.
\]

We can drop the absolute value signs because \(\sin \phi\) is never negative for \(0 \leq \phi \leq \pi\). Note that this is the same result we obtained in Section 15.6.

(Exercise 17). The corresponding version of Equation (7) is

\[
\iiint_{D} F(x, y, z) \, dx \, dy \, dz = \iiiint_{G} H(\rho, \phi, \theta) \, d\rho \, d\phi \, d\theta.
\]

Here is an example of another substitution. Although we could evaluate the integral in this example directly, we have chosen it to illustrate the substitution method in a simple (and fairly intuitive) setting.

**EXAMPLE 3**  
**Applying a Transformation to Integrate**

Evaluate

\[
\int_{0}^{1} \int_{0}^{1} \int_{x=y/2}^{x=(y/2)+1} \left( \frac{2x-y}{2} + \frac{z}{3} \right) \, dx \, dy \, dz
\]

by applying the transformation

\[
u = (2x - y)/2, \quad v = y/2, \quad w = z/3
\]

and integrating over an appropriate region in \(uvw\)-space.

**Solution**  
We sketch the region \(D\) of integration in \(xyz\)-space and identify its boundaries (Figure 15.54). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding \(uvw\)-region \(G\) and the Jacobian of the transformation. To find them, we first solve Equations (8) for \(x, y,\) and \(z\) in terms of \(u, v,\) and \(w\). Routine algebra gives

\[
x = u + v, \quad y = 2v, \quad z = 3w.
\]

We then find the boundaries of \(G\) by substituting these expressions into the equations for the boundaries of \(D\):
The goal of this section was to introduce you to the ideas involved in coordinate transformations. A thorough discussion of transformations, the Jacobian, and multivariable substitution is best given in an advanced calculus course after a study of linear algebra.

### 15.7 Substitutions in Multiple Integrals

<table>
<thead>
<tr>
<th>XYZ-equations for the boundary of $D$</th>
<th>Corresponding Uvw-equations for the boundary of $G$</th>
<th>Simplified Uvw-equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = y/2$</td>
<td>$u + v = 2v/2 = v$</td>
<td>$u = 0$</td>
</tr>
<tr>
<td>$x = (y/2) + 1$</td>
<td>$u + v = (2v/2) + 1 = v + 1$</td>
<td>$u = 1$</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$2u = 0$</td>
<td>$v = 0$</td>
</tr>
<tr>
<td>$y = 4$</td>
<td>$2v = 4$</td>
<td>$v = 2$</td>
</tr>
<tr>
<td>$z = 0$</td>
<td>$3w = 0$</td>
<td>$w = 0$</td>
</tr>
<tr>
<td>$z = 3$</td>
<td>$3w = 3$</td>
<td>$w = 1$</td>
</tr>
</tbody>
</table>

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$