Chapter 15: Multiple Integrals

Chapter 15 Additional and Advanced Exercises

Volumes

1. **Sand pile: double and triple integrals** The base of a sand pile covers the region in the xy-plane that is bounded by the parabola \( x^2 + y = 6 \) and the line \( y = x \). The height of the sand above the point \((x, y)\) is \( x^2 \). Express the volume of sand as (a) a double integral, (b) a triple integral. Then (c) find the volume.

2. **Water in a hemispherical bowl** A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.

3. **Solid cylindrical region between two planes** Find the volume of the portion of the solid cylinder \( x^2 + y^2 \leq 1 \) that lies between the planes \( z = 0 \) and \( x + y + z = 2 \).

4. **Sphere and paraboloid** Find the volume of the region bounded above by the sphere \( x^2 + y^2 + z^2 = 2 \) and below by the paraboloid \( z = x^2 + y^2 \).

5. **Two paraboloids** Find the volume of the region bounded above by the paraboloid \( z = 3 - x^2 - y^2 \) and below by the paraboloid \( z = 2x^2 + 2y^2 \).

6. **Spherical coordinates** Find the volume of the region enclosed by the spherical coordinate surface \( \rho = 2 \sin \phi \) (see accompanying figure).

7. **Hole in sphere** A circular cylindrical hole is bored through a solid sphere, the axis of the hole being a diameter of the sphere. The volume of the remaining solid is

\[
V = 2 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4r^2 - z^2}} r \, dr \, dz \, d\theta.
\]

a. Find the radius of the hole and the radius of the sphere.
b. Evaluate the integral.

8. **Sphere and cylinder** Find the volume of material cut from the solid sphere \( r^2 + z^2 \leq 9 \) by the cylinder \( r = 3 \sin \theta \).
9. Two paraboloids Find the volume of the region enclosed by the surfaces \( z = x^2 + y^2 \) and \( z = (x^2 + y^2 + 1)/2 \).

10. Cylinder and surface \( z = xy \) Find the volume of the region in the first octant that lies between the cylinders \( r = 1 \) and \( r = 2 \) and that is bounded below by the \( xy \)-plane and above by the surface \( z = xy \).

### Changing the Order of Integration

11. Evaluate the integral

\[
\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx.
\]

(Hint: Use the relation

\[
\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \int_a^b e^{-xy} \, dy
\]

to form a double integral and evaluate the integral by changing the order of integration.)

12. a. Polar coordinates Show, by changing to polar coordinates, that

\[
\int_0^{a \sin \beta} \int_{y \cos \beta}^{\sqrt{a^2 - y^2}} \ln (x^2 + y^2) \, dx \, dy = a^2 \beta \left( \ln a - \frac{1}{2} \right),
\]

where \( a > 0 \) and \( 0 < \beta < \pi/2 \).

b. Rewrite the Cartesian integral with the order of integration reversed.

13. Reducing a double to a single integral By changing the order of integration, show that the following double integral can be reduced to a single integral:

\[
\int_0^x \int_0^{u \sin \beta} f(t) \, dt \, du = \int_0^x (x - t) e^{u(x-t)} f(t) \, dt.
\]

Similarly, it can be shown that

\[
\int_0^x \int_0^{u \cos \beta} f(t) \, dt \, du = \int_0^x (x - t)^2 \frac{1}{2} e^{u(x-t)} f(t) \, dt.
\]

14. Transforming a double integral to obtain constant limits Sometimes a multiple integral with variable limits can be changed into one with constant limits. By changing the order of integration, show that

\[
\int_0^1 f(x) \left( \int_0^x g(x-y) f(y) \, dy \right) \, dx
\]

\[
= \int_0^1 f(y) \left( \int_y^1 g(x-y) f(x) \, dx \right) \, dy
\]

\[
= \frac{1}{2} \int_0^1 \int_0^1 g(|x - y|) f(x) f(y) \, dx \, dy.
\]

### Masses and Moments

15. Minimizing polar inertia A thin plate of constant density is to occupy the trapezoidal region in the first quadrant of the \( xy \)-plane having vertices \((0, 0), (a, 0), \) and \((a, 1/a)\). What value of \( a \) will minimize the plate’s polar moment of inertia about the origin?

16. Polar inertia of triangular plate Find the polar moment of inertia about the origin of a thin triangular plate of constant density \( \delta = 3 \) bounded by the \( y \)-axis and the lines \( y = 2x \) and \( y = 4 \) in the \( xy \)-plane.

17. Mass and polar inertia of a counterweight The counterweight of a flywheel of constant density 1 has the form of the smaller segment cut from a circle of radius \( a \) by a chord at a distance \( b \) from the center \((b < a)\). Find the mass of the counterweight and its polar moment of inertia about the center of the wheel.

18. Centroid of boomerang Find the centroid of the boomerang-shaped region between the parabolas \( y^2 = -4(x-1) \) and \( y^2 = 2x \) in the \( xy \)-plane.

### Theory and Applications

19. Evaluate

\[
\int_0^a \int_0^b e^{\max(bx^2, ay^2)} \, dy \, dx,
\]

where \( a \) and \( b \) are positive numbers and

\[
\max (bx^2, ay^2) = \begin{cases} 
 bx^2 & \text{if } bx^2 \geq ay^2 \\
 ay^2 & \text{if } bx^2 < ay^2.
\end{cases}
\]

20. Show that

\[
\iint \frac{\partial^2 F(x,y)}{dx dy} \, dx \, dy
\]

over the rectangle \( x_0 \leq x \leq x_1, y_0 \leq y \leq y_1 \), is

\[
F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).
\]

21. Suppose that \( f(x, y) \) can be written as a product \( f(x, y) = F(x)G(y) \) of a function of \( x \) and a function of \( y \). Then the integral of \( f \) over the rectangle \( R: a \leq x \leq b, c \leq y \leq d \) can be evaluated as a product as well, by the formula

\[
\iint_R f(x, y) \, dA = \left( \int_a^b F(x) \, dx \right) \left( \int_c^d G(y) \, dy \right).
\]
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22. Let \( D_a f \) denote the derivative of \( f(x, y) = (x^2 + y^2)/2 \) in the direction of the unit vector \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} \).

a. Finding average value Find the average value of \( D_a f \) over the triangular region cut from the first quadrant by the line \( x + y = 1 \).

b. Average value and centroid Show in general that the average value of \( D_a f \) over a region in the \( xy \)-plane is the value of \( D_a f \) at the centroid of the region.

23. The value of \( \Gamma(1/2) \) The gamma function,

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt,
\]

extends the factorial function from the nonnegative integers to other real values. Of particular interest in the theory of differential equations is the number

\[
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{(1/2)-1} e^{-t} \, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} \, dt.
\]

24. Total electrical charge over circular plate The electrical charge distribution on a circular plate of radius \( R \) meters is \( \sigma(r, \theta) = kr(1 - \sin \theta) \) coulomb/m\(^2\) (\( k \) a constant). Integrate \( \sigma \) over the plate to find the total charge \( Q \).

25. A parabolic rain gauge A bowl is in the shape of the graph of \( z = x^2 + y^2 \) from \( z = 0 \) to \( z = 10 \) in. You plan to calibrate the bowl to make it into a rain gauge. What height in the bowl would correspond to 1 in. of rain? 3 in. of rain?

26. Water in a satellite dish A parabolic satellite dish is 2 m wide and 1/2 m deep. Its axis of symmetry is tilted 30 degrees from the vertical.

a. Set up, but do not evaluate, a triple integral in rectangular coordinates that gives the amount of water the satellite dish will hold. (Hint: Put your coordinate system so that the satellite dish is in “standard position” and the plane of the water level is slanted.) (Caution: The limits of integration are not “nice.”)

b. What would be the smallest tilt of the satellite dish so that it holds no water?

27. An infinite half-cylinder Let \( D \) be the interior of the infinite right circular half-cylinder of radius 1 with its single-end face suspended 1 unit above the origin and its axis the ray from \((0, 0, 1)\) to \((1, 1, 1)\). Use cylindrical coordinates to evaluate

\[
\iiint_D z(r^2 + z^2)^{-5/2} \, dV.
\]

28. Hypervolume We have learned that \( \int_a^b 1 \, dx \) is the length of the interval \([a, b]\) on the number line (one-dimensional space), \( \iint_R 1 \, dA \) is the area of region \( R \) in the \( xy \)-plane (two-dimensional space), and \( \iiint_D 1 \, dV \) is the volume of the region \( D \) in three-dimensional space (\( xyz \)-space). We could continue: If \( Q \) is a region in 4-space (\( xyzw \)-space), then \( \iiint_Q 1 \, dV \) is the “hypervolume” of \( Q \). Use your generalizing abilities and a Cartesian coordinate system of 4-space to find the hypervolume inside the unit 4-sphere \( x^2 + y^2 + z^2 + w^2 = 1 \).