EXERCISES 16.3

Testing for Conservative Fields
Which fields in Exercises 1–6 are conservative, and which are not?
1. \( \mathbf{F} = yzi + xzj + xyk \)
2. \( \mathbf{F} = (y \sin z)i + (x \sin z)j + (xy \cos z)k \)
3. \( \mathbf{F} = yi + (x + z)j - yk \)
4. \( \mathbf{F} = -yi + xj \)
5. \( \mathbf{F} = (z + y)i + zj + (y + x)k \)
6. \( \mathbf{F} = (e^x \cos y)i - (e^x \sin y)j + zk \)

Finding Potential Functions
In Exercises 7–12, find a potential function \( f \) for the field \( \mathbf{F} \).
7. \( \mathbf{F} = 2xi + 3yj + 4k \)
8. \( \mathbf{F} = (y + z)i + (x + z)j + (x + y)k \)
9. \( \mathbf{F} = e^z(i + xj + 2k) \)
10. \( \mathbf{F} = (y \sin z)i + (x \sin z)j + (xy \cos z)k \)
11. \( \mathbf{F} = (\ln x + \sec^2(x + y))i + \frac{y}{y^2 + z^2}j + \frac{z}{y^2 + z^2}k \)
12. \( \mathbf{F} = \frac{y}{1 + x^2 y^2}i + \left( \frac{x}{1 + x^2 y^2} + \frac{z}{\sqrt{1 - y^2 z^2}} \right)j + \left( \frac{y}{\sqrt{1 - y^2 z^2}} + \frac{1}{y^2} \right)k \)

Evaluating Line Integrals
In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.
13. \( \int_{(0,0)}^{(2,3,-6)} 2x \, dx + 2y \, dy + 2z \, dz \)
14. \( \int_{(0,0)}^{(1,1,2)} yz \, dx + xz \, dy + xy \, dz \)
15. \( \int_{(0,0)}^{(1,2,3)} 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz \)
16. \( \int_{(0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1 + z^2} \, dz \)
17. \( \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz \)

Although they are not defined on all of space \( \mathbb{R}^3 \), the fields associated with Exercises 18–22 are simply connected and the Component Test can be used to show they are conservative. Find a potential function for each field and evaluate the integrals as in Example 4.
18. \( \int_{(0,2,1)}^{(1,3/2,2)} 2 \cos y \, dx + \left( \frac{1}{y} - 2x \sin y \right) \, dy + \frac{1}{z} \, dz \)

19. \( \int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz \)
20. \( \int_{(1,2,1)}^{(1,2,1)} (2x \ln y - yz) \, dx + \left( \frac{x^2}{y^2} - xz \right) \, dy - xy \, dz \)
21. \( \int_{(1,1,1)}^{(1,2,2)} \frac{1}{y} \, dx + \left( \frac{1}{x} - \frac{y}{z^2} \right) \, dy - \frac{y}{z} \, dz \)
22. \( \int_{(-1,-1,-1)}^{(2,3,-1)} 2x \, dx + 2y \, dy + 2z \, dz \)

23. Revisiting Example 4 Evaluate the integral
\( \int_{(1,1,1)}^{(1,2,3)} y \, dx + x \, dy + 4 \, dz \)

from Example 4 by finding parametric equations for the line segment from (1, 1, 1) to (2, 3, -1) and evaluating the line integral of \( \mathbf{F} = yi + xj + 4k \) along the segment. Since \( \mathbf{F} \) is conservative, the integral is independent of the path.

24. Evaluate
\( \int_{C} x^2 \, dx + yz \, dy + (y^2/2) \, dz \)
along the line segment \( C \) joining (0, 0, 0) to (0, 3, 4).

Theory, Applications, and Examples

Independence of path Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from \( A \) to \( B \).
25. \( \int_{A}^{B} z^2 \, dx + 2y \, dy + 2xz \, dz \)
26. \( \int_{A}^{B} x \, dx + y \, dy + z \, dz \)

In Exercises 27 and 28, find a potential function for \( \mathbf{F} \).
27. \( \mathbf{F} = \frac{2x}{y}i + \left( \frac{1 - x^2}{y^2} \right)j \)
28. \( \mathbf{F} = (e^x \ln y)i + \left( \frac{e^x}{y^2} + \sin z \right)j + (y \cos z)k \)

29. Work along different paths Find the work done by \( \mathbf{F} = (x^2 + y)i + (y^2 + x)j + ze^k \) over the following paths from (1, 0, 0) to (1, 0, 1).
a. The line segment \( x = 1, y = 0, 0 \leq z \leq 1 \)
b. The helix \( \mathbf{r}(t) = (\cos t)i + (\sin t)j + (t/2\pi)k, 0 \leq t \leq 2\pi \)
c. The x-axis from (1, 0, 0) to (0, 0, 0) followed by the parabola \( z = x^2, y = 0 \) from (0, 0, 0) to (1, 0, 1)

30. Work along different paths Find the work done by \( \mathbf{F} = e^{xz}i + (xze^{yz} + z \cos y)j + (yxe^{yz} + \sin y)k \) over the following paths from (1, 0, 1) to (1, π/2, 0).
16.3 Path Independence, Potential Functions, and Conservative Fields

31. Evaluating a work integral two ways

Let \( \mathbf{F} = \nabla (x^3 y^2) \) and let \( C \) be the path in the \( xy \)-plane from \((-1, 1)\) to \((1, 1)\) that consists of the line segment from \((-1, 1)\) to \((0, 0)\) followed by the line segment from \((0, 0)\) to \((1, 1)\). Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) in two ways.

a. Find parametrizations for the segments that make up \( C \) and evaluate the integral.

b. Using \( f(x, y) = x^3 y^2 \) as a potential function for \( \mathbf{F} \).

32. Integral along different paths

Evaluate \( \int_C 2x \cos y \, dx - x^2 \sin y \, dy \) along the following paths \( C \) in the \( xy \)-plane.

a. The parabola \( y = (x - 1)^2 \) from \((1, 0)\) to \((0, 1)\)

b. The line segment from \((-1, 0)\) to \((1, 0)\)

c. The \( x \)-axis from \((-1, 0)\) to \((1, 0)\)

d. The astroid \( r(t) = (\cos^3 t) \mathbf{i} + (\sin^3 t) \mathbf{j}, \quad 0 \leq t \leq 2\pi \)

clockwise from \((1, 0)\) back to \((1, 0)\)

33. a. Exact differential form

How are the constants \( a, b, \) and \( c \) related if the following differential form is exact?

\[
(a y^2 + 2c z x) \, dx + y (b x + c z) \, dy + (a y^2 + c x^2) \, dz
\]

b. Gradient field

For what values of \( b \) and \( c \) will

\[
\mathbf{F} = (y^2 + 2c z x) \mathbf{i} + y (b x + c z) \mathbf{j} + (a y^2 + c x^2) \mathbf{k}
\]

be a gradient field?

34. Gradient of a line integral

Suppose that \( \mathbf{F} = \nabla f \) is a conservative vector field and

\[
g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.
\]

Show that \( \nabla g = \mathbf{F} \).

35. Path of least work

You have been asked to find the path along which a force field \( \mathbf{F} \) will perform the least work in moving a particle between two locations. A quick calculation on your part shows \( \mathbf{F} \) to be conservative. How should you respond? Give reasons for your answer.

36. A revealing experiment

By experiment, you find that a force field \( \mathbf{F} \) performs only half as much work in moving an object along path \( C_1 \) from \( A \) to \( B \) as it does in moving the object along path \( C_2 \) from \( A \) to \( B \). What can you conclude about \( \mathbf{F} \)? Give reasons for your answer.

37. Work by a constant force

Show that the work done by a constant force field \( \mathbf{F} = ai + bj + ck \) in moving a particle along any path from \( A \) to \( B \) is \( W = \mathbf{F} \cdot \mathbf{AB} \).

38. Gravitational field

a. Find a potential function for the gravitational field

\[
\mathbf{F} = -G m M \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \quad (G, m, \text{ and } M \text{ are constants}).
\]

b. Let \( P_1 \) and \( P_2 \) be points at distance \( s_1 \) and \( s_2 \) from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from \( P_1 \) to \( P_2 \) is

\[
G m M \left( \frac{1}{s_2} - \frac{1}{s_1} \right) .
\]