A.2 Proofs of Limit Theorems

This appendix proves Theorem 1, Parts 2–5, and Theorem 4 from Section 2.2.

**THEOREM 1 Limit Laws**

If \( L, M, c, \) and \( k \) are real numbers and

- \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then
  1. **Sum Rule:** \( \lim_{x \to c} (f(x) + g(x)) = L + M \)
  2. **Difference Rule:** \( \lim_{x \to c} (f(x) - g(x)) = L - M \)
  3. **Product Rule:** \( \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M \)
  4. **Constant Multiple Rule:** \( \lim_{x \to c} (kf(x)) = kL \) (any number \( k \))
  5. **Quotient Rule:** \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \), if \( M \neq 0 \)
  6. **Power Rule:** If \( r \) and \( s \) are integers with no common factor and \( s \neq 0 \), then
     \[
     \lim_{x \to c} (f(x))^{r/s} = L^{r/s}
     \]
     provided that \( L^{r/s} \) is a real number. (If \( s \) is even, we assume that \( L > 0 \).)

We proved the Sum Rule in Section 2.3 and the Power Rule is proved in more advanced texts. We obtain the Difference Rule by replacing \( g(x) \) by \(-g(x)\) and \( M \) by \(-M\) in the Sum Rule. The Constant Multiple Rule is the special case \( g(x) = k \) of the Product Rule. This leaves only the Product and Quotient Rules.

**Proof of the Limit Product Rule** We show that for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x \) in the intersection \( D \) of the domains of \( f \) and \( g \),

\[
0 < |x - c| < \delta \Rightarrow |f(x)g(x) - LM| < \epsilon.
\]

Suppose then that \( \epsilon \) is a positive number, and write \( f(x) \) and \( g(x) \) as

\[
f(x) = L + (f(x) - L), \quad g(x) = M + (g(x) - M).
\]
Multiply these expressions together and subtract $LM$:

\[ f(x) \cdot g(x) - LM = (L + (f(x) - L))(M + (g(x) - M)) - LM \]
\[ = LM + L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M) \]
\[ = L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M). \quad (1) \]

Since $f$ and $g$ have limits $L$ and $M$ as $x \to c$, there exist positive numbers $\delta_1, \delta_2, \delta_3$, and $\delta_4$ such that for all $x$ in $D$

\[
\begin{align*}
0 < |x - c| < \delta_1 & \implies |f(x) - L| < \sqrt{\epsilon/3} \\
0 < |x - c| < \delta_2 & \implies |g(x) - M| < \sqrt{\epsilon/3} \\
0 < |x - c| < \delta_3 & \implies |f(x) - L| < \epsilon/(3(1 + |M|)) \\
0 < |x - c| < \delta_4 & \implies |g(x) - M| < \epsilon/(3(1 + |L|)).
\end{align*} \tag{2}
\]

If we take $\delta$ to be the smallest numbers $\delta_1$ through $\delta_4$, the inequalities on the right-hand side of the implications (2) will hold simultaneously for $0 < |x - c| < \delta$. Therefore, for all $x$ in $D$, $0 < |x - c| < \delta$ implies

\[ |f(x) \cdot g(x) - LM| \leq |L||g(x) - M| + |M||f(x) - L| + |f(x) - L||g(x) - M| \]
\[ \leq (1 + |L|)|g(x) - M| + (1 + |M|)|f(x) - L| + |f(x) - L||g(x) - M| \]
\[ < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sqrt{\frac{\epsilon}{3}} \cdot \sqrt{\frac{\epsilon}{3}} = \epsilon. \quad \text{Values from (2)} \]

This completes the proof of the Limit Product Rule.

**Proof of the Limit Quotient Rule** We show that $\lim_{x \to c} (1/g(x)) = 1/M$. We can then conclude that

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \left( f(x) \cdot \frac{1}{g(x)} \right) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}
\]

by the Limit Product Rule.

Let $\epsilon > 0$ be given. To show that $\lim_{x \to c} (1/g(x)) = 1/M$, we need to show that there exists a $\delta > 0$ such that for all $x$

\[ 0 < |x - c| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \epsilon. \]

Since $|M| > 0$, there exists a positive number $\delta_1$ such that for all $x$

\[ 0 < |x - c| < \delta_1 \implies |g(x) - M| < \frac{M}{2}. \quad (3) \]

For any numbers $A$ and $B$ it can be shown that $|A| - |B| \leq |A - B|$ and $|B| - |A| \leq |A - B|$, from which it follows that $|A| - |B| \leq |A - B|$. With $A = g(x)$ and $B = M$, this becomes

\[ |g(x)| - |M| \leq |g(x) - M|, \]

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which can be combined with the inequality on the right in Implication (3) to get, in turn,

\[
\frac{|M|}{2} < |g(x)| - |M| < \frac{|M|}{2}
\]

Therefore, implies that

\[
\frac{1}{|g(x)|} < \frac{2}{|M|} < \frac{3}{|g(x)|}
\]

(4)

Therefore, \(0 < |x - c| < \delta_1\) implies that

\[
\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{Mg(x)} \right| \leq \frac{1}{|M|} \cdot \frac{1}{|g(x)|} \cdot |M - g(x)|
\]

\[
< \frac{1}{|M|} \cdot \frac{2}{|M|} \cdot |M - g(x)|. \quad \text{Inequality (4)}
\]

(5)

Since \((1/2)|M|^2 \varepsilon > 0\), there exists a number \(\delta_2 > 0\) such that for all \(x\)

\[
0 < |x - c| < \delta_2 \implies |M - g(x)| < \frac{\varepsilon}{2} |M|^2.
\]

(6)

If we take \(\delta\) to be the smaller of \(\delta_1\) and \(\delta_2\), the conclusions in (5) and (6) both hold for all \(x\) such that \(0 < |x - c| < \delta\). Combining these conclusions gives

\[
0 < |x - c| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon.
\]

This concludes the proof of the Limit Quotient Rule.

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**THEOREM 4  The Sandwich Theorem**

Suppose that \(g(x) \leq f(x) \leq h(x)\) for all \(x\) in some open interval \(I\) containing \(c\), except possibly at \(x = c\) itself. Suppose also that \(\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L\). Then \(\lim_{x \to c} f(x) = L\).

**Proof for Right-Hand Limits** Suppose \(\lim_{x \to c^+} g(x) = \lim_{x \to c^+} h(x) = L\). Then for any \(\varepsilon > 0\) there exists a \(\delta > 0\) such that for all \(x\) the interval \(c < x < c + \delta\) is contained in \(I\) and the inequality implies

\[
L - \varepsilon < g(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < h(x) < L + \varepsilon.
\]

These inequalities combine with the inequality \(g(x) \leq f(x) \leq h(x)\) to give

\[
L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon,
\]

\[
L - \varepsilon < f(x) < L + \varepsilon,
\]

\[
- \varepsilon < f(x) - L < \varepsilon.
\]

Therefore, for all \(x\), the inequality \(c < x < c + \delta\) implies \(|f(x) - L| < \varepsilon\).
A.2 Proofs of Limit Theorems

**Proof for Two-Sided Limits** If \( \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \), then \( g(x) \) and \( h(x) \) both approach \( L \) as \( x \to c^+ \) and as \( x \to c^- \); so \( \lim_{x \to c^+} f(x) = L \) and \( \lim_{x \to c^-} f(x) = L \). Hence \( \lim_{x \to c} f(x) \) exists and equals \( L \).

**EXERCISES A.2**

1. Suppose that functions \( f_1(x), f_2(x), \) and \( f_3(x) \) have limits \( L_1, L_2, \) and \( L_3 \), respectively, as \( x \to c \). Show that their sum has limit \( L_1 + L_2 + L_3 \). Use mathematical induction (Appendix 1) to generalize this result to the sum of any finite number of functions.

2. Use mathematical induction and the Limit Product Rule in Theorem 1 to show that if functions \( f_1(x), f_2(x), \ldots, f_n(x) \) have limits \( L_1, L_2, \ldots, L_n \) as \( x \to c \), then
   \[
   \lim_{x \to c} f_1(x)f_2(x) \cdots f_n(x) = L_1 \cdot L_2 \cdots \cdot L_n.
   \]

3. Use the fact that \( \lim_{x \to c^+} x = c \) and the result of Exercise 2 to show that \( \lim_{x \to c^-} x^n = c^n \) for any integer \( n > 1 \).

4. Limits of polynomials Use the fact that \( \lim_{x \to c^+}(k) = k \) for any number \( k \) together with the results of Exercises 1 and 3 to show that \( \lim_{x \to c^-} f(x) = f(c) \) for any polynomial function \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \).

5. Limits of rational functions Use Theorem 1 and the result of Exercise 4 to show that if \( f(x) \) and \( g(x) \) are polynomial functions and \( g(c) \neq 0 \), then
   \[
   \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.
   \]

6. Composites of continuous functions Figure A.1 gives the diagram for a proof that the composite of two continuous functions is continuous. Reconstruct the proof from the diagram. The statement to be proved is this: If \( f \) is continuous at \( x = c \) and \( g \) is continuous at \( f(c) \), then \( g \circ f \) is continuous at \( c \).

Assume that \( c \) is an interior point of the domain of \( f \) and that \( f(c) \) is an interior point of the domain of \( g \). This will make the limits involved two-sided. (The arguments for the cases that involve one-sided limits are similar.)

![Figure A.1](image-url) The diagram for a proof that the composite of two continuous functions is continuous.