A.3 Commonly Occurring Limits

This appendix verifies limits (4)–(6) in Theorem 5 of Section 11.1.

**Limit 4: If** \(|x| < 1\), \(\lim_{n \to \infty} x^n = 0\) **We need to show that to each** \(\epsilon > 0\) **there corresponds an integer** \(N\) **so large that** \(|x^n| < \epsilon\) **for all** \(n > N\). **Since** \(\epsilon^{1/n} \to 1\), **while**...
Let \( |x| < 1 \), there exists an integer \( N \) for which \( \epsilon \) 1/\( N \) > |\( x | \|. In other words,

\[
|x|^N = |x|^N < \epsilon.
\]

This is the integer we seek because, if \( |x| < 1 \), then

\[
|x|^n < |x|^N \quad \text{for all } n > N.
\]

Combining (1) and (2) produces \( |x|^n < \epsilon \) for all \( n > N \), concluding the proof.

Limit 5: For any number \( x \), \( \lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x \). Let

\[
a_n = \left(1 + \frac{x}{n}\right)^n.
\]

Then

\[
\ln a_n = \ln \left(1 + \frac{x}{n}\right)^n = n \ln \left(1 + \frac{x}{n}\right) \to x,
\]

as we can see by the following application of l'Hôpital's Rule, in which we differentiate with respect to \( n \):

\[
\lim_{n \to \infty} \frac{\ln(1 + x/n)}{1/n} = \lim_{n \to \infty} \frac{1}{1 + x/n} \cdot \frac{-x}{n^2} = \lim_{n \to \infty} \frac{x}{1 + x/n} = x.
\]

Apply Theorem 4, Section 11.1, with \( f(x) = e^x \) to conclude that

\[
\left(1 + \frac{x}{n}\right)^n = a_n = e^{\ln a_n} \to e^x.
\]

Limit 6: For any number \( x \), \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \). Since

\[
-\frac{|x|^n}{n!} \leq \frac{x^n}{n!} \leq \frac{|x|^n}{n!},
\]

all we need to show is that \( |x|^n/n! \to 0 \). We can then apply the Sandwich Theorem for Sequences (Section 11.1, Theorem 2) to conclude that \( x^n/n! \to 0 \).

The first step in showing that \( |x|^n/n! \to 0 \) is to choose an integer \( M > |x| \), so that \( (|x|/M) < 1 \). By Limit 4, just proved, we then have \( (|x|/M)^n \to 0 \). We then restrict our attention to values of \( n > M \). For these values of \( n \), we can write

\[
\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdot \cdots \cdot M \cdot (M + 1)(M + 2) \cdots n}
\]

\[
\leq \frac{|x|^n}{M!M^{n-M}} = \frac{|x|^nM^M}{M!M^n} = \frac{M^M}{M!} \left(\frac{|x|}{M}\right)^n.
\]
Thus, 

\[ 0 \leq \frac{|x|^n}{n!} \leq \frac{M^M}{M!} \left(\frac{|x|}{M}\right)^n. \]

Now, the constant \(M^M/M!\) does not change as \(n\) increases. Thus the Sandwich Theorem tells us that \(|x|^n/n! \to 0\) because \((|x|/M)^n \to 0\).